Chapter 7 Applications of Definite Integrals

The art of pottery developed independently in many ancient civilizations and still exists in modern times. The desired shape of the side of a pottery vase can be described by:

$$y = 5.0 + 2 \sin (x/4) \quad (0 \leq x \leq 8\pi)$$

where x is the height and y is the radius at height x (in inches).

A base for the vase is preformed and placed on a potter's wheel. How much clay should be added to the base to form this vase if the inside radius is always 1 inch less than the outside radius? Section 7.3 contains the needed mathematics.

Chapter 7 Overview

By this point it should be apparent that finding the limits of Riemann sums is not just an intellectual exercise; it is a natural way to calculate mathematical or physical quantities that appear to be irregular when viewed as a whole, but which can be fragmented into regular pieces. We calculate values for the regular pieces using known formulas, then sum them to find a value for the irregular whole. This approach to problem solving was around for thousands of years before calculus came along, but it was tedious work and the more accurate you wanted to be the more tedious it became.

With calculus it became possible to get *exact* answers for these problems with almost no effort, because in the limit these sums became definite integrals and definite integrals could be evaluated with antiderivatives. With calculus, the challenge became one of fitting an integrable function to the situation at hand (the "modeling" step) and then finding an antiderivative for it.

Today we can finesse the antidifferentiation step (occasionally an insurmountable hurdle for our predecessors) with programs like NINT, but the modeling step is no less crucial. Ironically, it is the modeling step that is thousands of years old. Before either calculus or technology can be of assistance, we must still break down the irregular whole into regular parts and set up a function to be integrated. We have already seen how the process works with area, volume, and average value, for example. Now we will focus more closely on the underlying modeling step: how to set up the function to be integrated.

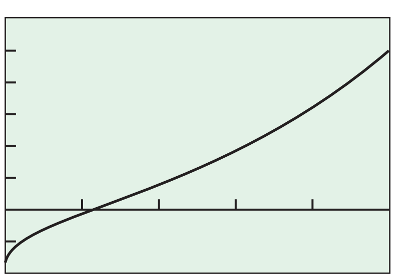
7.1 Integral As Net Change

What you'll learn about

- Linear Motion Revisited
- General Strategy
- Consumption Over Time
- Net Change from Data
- Work

. . . and why

The integral is a tool that can be used to calculate net change and total accumulation.



[0, 5] by [–10, 30]

Figure 7.1 The velocity function in Example 1.

Linear Motion Revisited

In many applications, the integral is viewed as net change over time. The classic example of this kind is distance traveled, a problem we discussed in Chapter 5.

EXAMPLE 1 Interpreting a Velocity Function

Figure 7.1 shows the velocity

$$\frac{ds}{dt} = v(t) = t^2 - \frac{8}{(t+1)^2} \qquad \frac{\text{cm}}{\text{sec}}$$

of a particle moving along a horizontal s-axis for $0 \le t \le 5$. Describe the motion.

SOLUTION

Solve Graphically The graph of v (Figure 7.1) starts with $v(0) = -8$, which we interpret as saying that the particle has an initial velocity of 8 cm/sec to the left. It slows to a halt at about $t = 1.25$ sec, after which it moves to the right ($v > 0$) with increasing speed, reaching a velocity of $v(5) \approx 24.8$ cm/sec at the end. ***Now try Exercise 1(a).***

EXAMPLE 2 Finding Position from Displacement

Suppose the initial position of the particle in Example 1 is $s(0) = 9$. What is the particle's position at **(a)** $t = 1$ sec? **(b)** $t = 5$ sec?

SOLUTION

Solve Analytically

(a) The position at $t = 1$ is the initial position $s(0)$ plus the displacement (the amount, Δs, that the position changed from $t = 0$ to $t = 1$). When velocity is

continued

Reminder from Section 3.4

A change in position is a **displacement.** If $s(t)$ is a body's position at time t, the displacement over the time interval from t to $t + \Delta t$ is $s(t + \Delta t) - s(t)$. The displacement may be positive, negative, or zero, depending on the motion.

constant during a motion, we can find the displacement (change in position) with the formula

$$\text{Displacement} = \text{rate of change} \times \text{time}.$$

But in our case the velocity varies, so we resort instead to partitioning the time interval $[0, 1]$ into subintervals of length Δt so short that the velocity is effectively constant on each subinterval. If t_k is any time in the kth subinterval, the particle's velocity throughout that interval will be close to $v(t_k)$. The change in the particle's position during the brief time this constant velocity applies is

$$v(t_k)\,\Delta t. \qquad \text{rate of change} \times \text{time}$$

If $v(t_k)$ is negative, the displacement is negative and the particle will move left. If $v(t_k)$ is positive, the particle will move right. The sum

$$\sum v(t_k)\,\Delta t$$

of all these small position changes approximates the displacement for the time interval $[0, 1]$.

The sum $\sum v(t_k)\,\Delta t$ is a Riemann sum for the continuous function $v(t)$ over $[0, 1]$. As the norms of the partitions go to zero, the approximations improve and the sums converge to the integral of v over $[0, 1]$, giving

$$\begin{aligned}\text{Displacement} &= \int_0^1 v(t)\,dt \\ &= \int_0^1 \left(t^2 - \frac{8}{(t+1)^2}\right) dt \\ &= \left[\frac{t^3}{3} + \frac{8}{t+1}\right]_0^1 \\ &= \frac{1}{3} + \frac{8}{2} - 8 = -\frac{11}{3}.\end{aligned}$$

During the first second of motion, the particle moves $11/3$ cm to the left. It starts at $s(0) = 9$, so its position at $t = 1$ is

$$\text{New position} = \text{initial position} + \text{displacement} = 9 - \frac{11}{3} = \frac{16}{3}.$$

(b) If we model the displacement from $t = 0$ to $t = 5$ in the same way, we arrive at

$$\text{Displacement} = \int_0^5 v(t)\,dt = \left[\frac{t^3}{3} + \frac{8}{t+1}\right]_0^5 = 35.$$

The motion has the net effect of displacing the particle 35 cm to the right of its starting point. The particle's final position is

$$\begin{aligned}\text{Final position} &= \text{initial position} + \text{displacement} \\ &= s(0) + 35 = 9 + 35 = 44.\end{aligned}$$

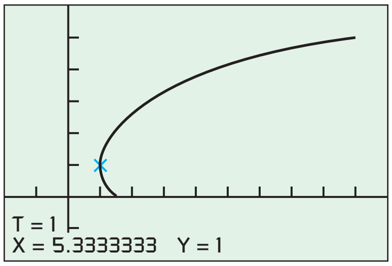
Support Graphically The position of the particle at any time t is given by

$$s(t) = \int_0^t \left[u^2 - \frac{8}{(u+1)^2}\right] du + 9,$$

because $s'(t) = v(t)$ and $s(0) = 9$. Figure 7.2 shows the graph of $s(t)$ given by the parametrization

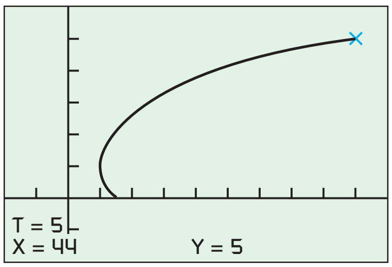
$$x(t) = \text{NINT}\,(v(u), u, 0, t) + 9, \qquad y(t) = t, \qquad 0 \le t \le 5.$$

continued



[–10, 50] by [–2, 6]

(a)



[–10, 50] by [–2, 6]

(b)

Figure 7.2 Using TRACE and the parametrization in Example 2 you can "see" the left and right motion of the particle.

(a) Figure 7.2a supports that the position of the particle at $t = 1$ is 16/3.

(b) Figure 7.2b shows the position of the particle is 44 at $t = 5$. Therefore, the displacement is $44 - 9 = 35$. ***Now try Exercise 1(b).***

The reason for our method in Example 2 was to illustrate the *modeling step* that will be used throughout this chapter. We can also solve Example 2 using the techniques of Chapter 6 as shown in Exploration 1.

EXPLORATION 1 Revisiting Example 2

The velocity of a particle moving along a horizontal s-axis for $0 \le t \le 5$ is

$$\frac{ds}{dt} = t^2 - \frac{8}{(t+1)^2}.$$

1. Use the indefinite integral of ds/dt to find the solution of the initial value problem

$$\frac{ds}{dt} = t^2 - \frac{8}{(t+1)^2}, \qquad s(0) = 9.$$

2. Determine the position of the particle at $t = 1$. Compare your answer with the answer to Example 2a.
3. Determine the position of the particle at $t = 5$. Compare your answer with the answer to Example 2b.

We know now that the particle in Example 1 was at $s(0) = 9$ at the beginning of the motion and at $s(5) = 44$ at the end. But it did not travel from 9 to 44 directly—it began its trip by moving to the left (Figure 7.2). How much distance did the particle actually travel? We find out in Example 3.

EXAMPLE 3 Calculating Total Distance Traveled

Find the *total distance traveled* by the particle in Example 1.

SOLUTION

Solve Analytically We partition the time interval as in Example 2 but record every position shift as *positive* by taking absolute values. The Riemann sum approximating total distance traveled is

$$\sum |v(t_k)|\,\Delta t,$$

and we are led to the integral

$$\text{Total distance traveled} = \int_0^5 |v(t)|\,dt = \int_0^5 \left| t^2 - \frac{8}{(t+1)^2} \right| dt.$$

Evaluate Numerically We have

$$\text{NINT}\left(\left| t^2 - \frac{8}{(t+1)^2} \right|, t, 0, 5 \right) \approx 42.59.$$

Now try Exercise 1(c).

What we learn from Examples 2 and 3 is this: Integrating velocity gives displacement (net area between the velocity curve and the time axis). Integrating the *absolute value* of velocity gives total distance traveled (total area between the velocity curve and the time axis).

General Strategy

The idea of fragmenting net effects into finite sums of easily estimated small changes is not new. We used it in Section 5.1 to estimate cardiac output, volume, and air pollution. What *is* new is that we can now identify many of these sums as Riemann sums and express their limits as integrals. The advantages of doing so are twofold. First, we can evaluate one of these integrals to get an accurate result in less time than it takes to crank out even the crudest estimate from a finite sum. Second, the integral itself becomes a formula that enables us to solve similar problems without having to repeat the modeling step.

The strategy that we began in Section 5.1 and have continued here is the following:

Strategy for Modeling with Integrals

1. *Approximate what you want to find as a Riemann sum* of values of a continuous function multiplied by interval lengths. If $f(x)$ is the function and $[a, b]$ the interval, and you partition the interval into subintervals of length Δx, the approximating sums will have the form $\sum f(c_k)\,\Delta x$ with c_k a point in the kth subinterval.
2. *Write a definite integral,* here $\int_a^b f(x)\,dx$, to express the limit of these sums as the norms of the partitions go to zero.
3. *Evaluate the integral* numerically or with an antiderivative.

EXAMPLE 4 Modeling the Effects of Acceleration

A car moving with initial velocity of 5 mph accelerates at the rate of $a(t) = 2.4t$ mph per second for 8 seconds.

(a) How fast is the car going when the 8 seconds are up?

(b) How far did the car travel during those 8 seconds?

SOLUTION

(a) We first model the effect of the acceleration on the car's velocity.

Step 1: *Approximate the net change in velocity as a Riemann sum.* When acceleration is constant,

$$\text{velocity change} = \text{acceleration} \times \text{time applied}.$$

rate of change × time

To apply this formula, we partition $[0, 8]$ into short subintervals of length Δt. On each subinterval the acceleration is nearly constant, so if t_k is any point in the kth subinterval, the change in velocity imparted by the acceleration in the subinterval is approximately

$$a(t_k)\,\Delta t \text{ mph.}$$

$\frac{\text{mph}}{\text{sec}} \times \text{sec}$

The net change in velocity for $0 \le t \le 8$ is approximately

$$\sum a(t_k)\,\Delta t \text{ mph.}$$

Step 2: *Write a definite integral.* The limit of these sums as the norms of the partitions go to zero is

$$\int_0^8 a(t)\,dt.$$

continued

Step 3: *Evaluate the integral.* Using an antiderivative, we have

$$\text{Net velocity change} = \int_0^8 2.4t\,dt = 1.2t^2\Big]_0^8 = 76.8 \text{ mph.}$$

So, how fast is the car going when the 8 seconds are up? Its initial velocity is 5 mph and the acceleration adds another 76.8 mph for a total of 81.8 mph.

(b) There is nothing special about the upper limit 8 in the preceding calculation. Applying the acceleration for any length of time t adds

$$\int_0^t 2.4u\,du \text{ mph}$$

u is just a dummy variable here.

to the car's velocity, giving

$$v(t) = 5 + \int_0^t 2.4u\,du = 5 + 1.2t^2 \text{ mph.}$$

The distance traveled from $t = 0$ to $t = 8$ sec is

$$\begin{aligned}\int_0^8 |v(t)|\,dt &= \int_0^8 (5 + 1.2t^2)\,dt \\ &= \Big[5t + 0.4t^3\Big]_0^8 \\ &= 244.8 \text{ mph} \times \text{seconds.}\end{aligned}$$

Extension of Example 3

Miles-per-hour second is not a distance unit that we normally work with! To convert to miles we multiply by hours/second $= 1/3600$, obtaining

$$244.8 \times \frac{1}{3600} = 0.068 \text{ mile.}$$

$\frac{\text{mi}}{\text{h}} \times \text{sec} \times \frac{\text{h}}{\text{sec}} = \text{mi}$

The car traveled 0.068 mi during the 8 seconds of acceleration. ***Now try Exercise 9.***

Consumption Over Time

The integral is a natural tool to calculate net change and total accumulation of more quantities than just distance and velocity. Integrals can be used to calculate growth, decay, and, as in the next example, consumption. Whenever you want to find the cumulative effect of a varying rate of change, integrate it.

EXAMPLE 5 Potato Consumption

From 1970 to 1980, the rate of potato consumption in a particular country was $C(t) = 2.2 + 1.1^t$ millions of bushels per year, with t being years since the beginning of 1970. How many bushels were consumed from the beginning of 1972 to the end of 1973?

SOLUTION

We seek the cumulative effect of the consumption rate for $2 \le t \le 4$.

Step 1: *Riemann sum.* We partition [2, 4] into subintervals of length Δt and let t_k be a time in the kth subinterval. The amount consumed during this interval is approximately

$$C(t_k)\,\Delta t \text{ million bushels.}$$

The consumption for $2 \le t \le 4$ is approximately

$$\sum C(t_k)\,\Delta t \text{ million bushels.}$$

continued

Step 2: *Definite integral.* The amount consumed from $t = 2$ to $t = 4$ is the limit of these sums as the norms of the partitions go to zero.

$$\int_2^4 C(t)\,dt = \int_2^4 (2.2 + 1.1^t)\,dt \text{ million bushels}$$

Step 3: *Evaluate.* Evaluating numerically, we obtain

$$\text{NINT}\,(2.2 + 1.1^t, t, 2, 4) \approx 7.066 \text{ million bushels.}$$

Now try Exercise 21.

Net Change from Data

Many real applications begin with data, not a fully modeled function. In the next example, we are given data on the rate at which a pump operates in consecutive 5-minute intervals and asked to find the total amount pumped.

Table 7.1 Pumping Rates

Time (min)	Rate (gal/min)
0	58
5	60
10	65
15	64
20	58
25	57
30	55
35	55
40	59
45	60
50	60
55	63
60	63

EXAMPLE 6 Finding Gallons Pumped from Rate Data

A pump connected to a generator operates at a varying rate, depending on how much power is being drawn from the generator to operate other machinery. The rate (gallons per minute) at which the pump operates is recorded at 5-minute intervals for one hour as shown in Table 7.1. How many gallons were pumped during that hour?

SOLUTION

Let $R(t)$, $0 \le t \le 60$, be the pumping rate as a continuous function of time for the hour. We can partition the hour into short subintervals of length Δt on which the rate is nearly constant and form the sum $\sum R(t_k)\,\Delta t$ as an approximation to the amount pumped during the hour. This reveals the integral formula for the number of gallons pumped to be

$$\text{Gallons pumped} = \int_0^{60} R(t)\,dt.$$

We have no formula for R in this instance, but the 13 equally spaced values in Table 7.1 enable us to estimate the integral with the Trapezoidal Rule:

$$\int_0^{60} R(t)\,dt \approx \frac{60}{2 \cdot 12}\left[58 + 2(60) + 2(65) + \cdots + 2(63) + 63\right]$$
$$= 3582.5.$$

The total amount pumped during the hour is about 3580 gal. *Now try Exercise 27.*

Joules

The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

1 joule = (1 newton)(1 meter).

In symbols, 1 J = 1 N • m.

It takes a force of about 1 N to lift an apple from a table. If you lift it 1 m you have done about 1 J of work on the apple. If you eat the apple, you will have consumed about 80 food calories, the heat equivalent of nearly 335,000 joules. If this energy were directly useful for mechanical work (it's not), it would enable you to lift 335,000 more apples up 1 m.

Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. When a body moves a distance d along a straight line as a result of the action of a force of constant magnitude F in the direction of motion, the **work** done by the force is

$$W = Fd.$$

The equation $W = Fd$ is the **constant-force formula** for work.

The units of work are force × distance. In the metric system, the unit is the newton-meter, which, for historical reasons, is called a joule (see margin note). In the U.S. customary system, the most common unit of work is the **foot-pound.**

Hooke's Law for springs says that the force it takes to stretch or compress a spring x units from its natural (unstressed) length is a constant times x. In symbols,

$$F = kx,$$

where k, measured in force units per unit length, is a characteristic of the spring called the **force constant.**

EXAMPLE 7 A Bit of Work

It takes a force of 10 N to stretch a spring 2 m beyond its natural length. How much work is done in stretching the spring 4 m from its natural length?

SOLUTION

We let $F(x)$ represent the force in newtons required to stretch the spring x meters from its natural length. By Hooke's Law, $F(x) = kx$ for some constant k. We are told that

$$F(2) = 10 = k \cdot 2,$$

The force required to stretch the spring 2 m is 10 newtons.

so $k = 5$ N/m and $F(x) = 5x$ for this particular spring.

We construct an integral for the work done in applying F over the interval from $x = 0$ to $x = 4$.

Step 1: *Riemann sum.* We partition the interval into subintervals on each of which F is so nearly constant that we can apply the constant-force formula for work. If x_k is any point in the kth subinterval, the value of F throughout the interval is approximately $F(x_k) = 5x_k$. The work done by F across the interval is approximately $5x_k\,\Delta x$, where Δx is the length of the interval. The sum

$$\sum F(x_k)\,\Delta x = \sum 5x_k\,\Delta x$$

approximates the work done by F from $x = 0$ to $x = 4$.

Steps 2 and 3: *Integrate.* The limit of these sums as the norms of the partitions go to zero is

$$\int_0^4 F(x)\,dx = \int_0^4 5x\,dx = 5\frac{x^2}{2}\Bigg]_0^4 = 40 \text{ N} \cdot \text{m}.$$

Numerically, work is the area under the force graph.

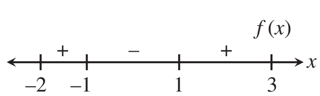
Now try Exercise 29.

We will revisit work in Section 7.5.

Quick Review 7.1 *(For help, go to Section 1.2.)*

In Exercises 1–10, find all values of x (if any) at which the function changes sign on the given interval. Sketch a number line graph of the interval, and indicate the sign of the function on each subinterval.

Example: $f(x) = x^2 - 1$ on $[-2, 3]$



Changes sign at $x = \pm 1$.

1. $\sin 2x$ on $[-3, 2]$ Changes sign at $-\frac{\pi}{2}, 0, \frac{\pi}{2}$
2. $x^2 - 3x + 2$ on $[-2, 4]$ Changes sign at 1, 2
3. $x^2 - 2x + 3$ on $[-4, 2]$ Always positive
4. $2x^3 - 3x^2 + 1$ on $[-2, 2]$ Changes sign at $-\frac{1}{2}$
5. $x \cos 2x$ on $[0, 4]$ Changes sign at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$
6. xe^{-x} on $[0, \infty)$ Always positive
7. $\frac{x}{x^2 + 1}$ on $[-5, 30]$ Changes sign at 0
8. $\frac{x^2 - 2}{x^2 - 4}$ on $[-3, 3]$ Changes sign at $-2, -\sqrt{2}, \sqrt{2}, 2$
9. $\sec(1 + \sqrt{1 - \sin^2 x})$ on $(-\infty, \infty)$
10. $\sin(1/x)$ on $[0.1, 0.2]$ Changes sign at $\frac{1}{3\pi}, \frac{1}{2\pi}$

9. Changes sign at $0.9633 + k\pi$, $2.1783 + k\pi$, where k is an integer

Section 7.1 Exercises

In Exercises 1–8, the function $v(t)$ is the velocity in m/sec of a particle moving along the x-axis. Use analytic methods to do each of the following:

(a) Determine when the particle is moving to the right, to the left, and stopped.

(b) Find the particle's displacement for the given time interval. If $s(0) = 3$, what is the particle's final position?

(c) Find the total distance traveled by the particle.

1. $v(t) = 5\cos t, \quad 0 \le t \le 2\pi$ See page 389.

2. $v(t) = 6\sin 3t, \quad 0 \le t \le \pi/2$ See page 389.

3. $v(t) = 49 - 9.8t, \quad 0 \le t \le 10$ See page 389.

4. $v(t) = 6t^2 - 18t + 12, \quad 0 \le t \le 2$ See page 389.

5. $v(t) = 5\sin^2 t\cos t, \quad 0 \le t \le 2\pi$ See page 389.

6. $v(t) = \sqrt{4 - t}, \quad 0 \le t \le 4$ See page 389.

7. $v(t) = e^{\sin t}\cos t, \quad 0 \le t \le 2\pi$ See page 389.

8. $v(t) = \dfrac{t}{1 + t^2}, \quad 0 \le t \le 3$ See page 389.

9. An automobile accelerates from rest at $1 + 3\sqrt{t}$ mph/sec for 9 seconds.

(a) What is its velocity after 9 seconds? 63 mph

(b) How far does it travel in those 9 seconds? 344.52 feet

10. A particle travels with velocity

$$v(t) = (t - 2)\sin t \ \text{ m/sec}$$

for $0 \le t \le 4$ sec.

(a) What is the particle's displacement? ≈ -1.44952 meters

(b) What is the total distance traveled? ≈ 1.91411 meters

11. ***Projectile*** Recall that the acceleration due to Earth's gravity is 32 ft/sec^2. From ground level, a projectile is fired straight upward with velocity 90 feet per second.

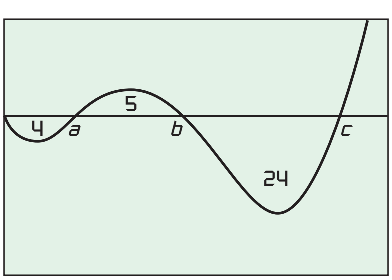
(a) What is its velocity after 3 seconds? -6 ft/sec

(b) When does it hit the ground? 5.625 sec

(c) When it hits the ground, what is the net distance it has traveled? 0

(d) When it hits the ground, what is the total distance it has traveled? 253.125 feet

In Exercises 12–16, a particle moves along the x-axis (units in cm). Its initial position at $t = 0$ sec is $x(0) = 15$. The figure shows the graph of the particle's velocity $v(t)$. The numbers are the *areas* of the enclosed regions.



12. What is the particle's displacement between $t = 0$ and $t = c$? -23 cm

13. What is the total distance traveled by the particle in the same time period? 33 cm

14. Give the positions of the particle at times a, b, and c. a: 11 b: 16 c: -8

15. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, b]$? $t = a$

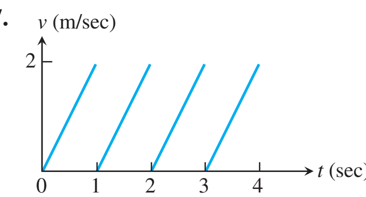
16. Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, c]$? $t = c$

In Exercises 17–20, the graph of the velocity of a particle moving on the x-axis is given. The particle starts at $x = 2$ when $t = 0$.

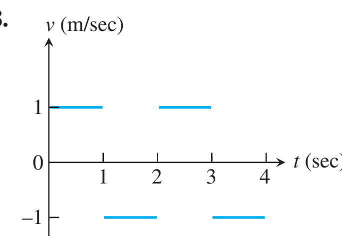
(a) Find where the particle is at the end of the trip.

(b) Find the total distance traveled by the particle.

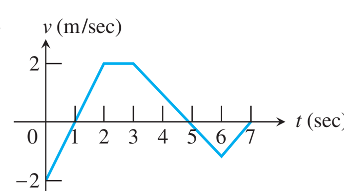
17. **(a)** 6 **(b)** 4 meters



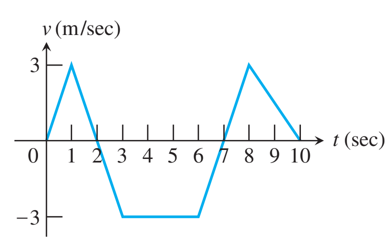
18. **(a)** 2 **(b)** 4 meters



19. **(a)** 5 **(b)** 7 meters



20. **(a)** -2.5 **(b)** 19.5 meters



21. ***U.S. Oil Consumption*** The rate of consumption of oil in the United States during the 1980s (in billions of barrels per year) is modeled by the function $C = 27.08 \cdot e^{t/25}$, where t is the number of years after January 1, 1980. Find the total consumption of oil in the United States from January 1, 1980 to January 1, 1990. ≈ 332.965 billion barrels

22. ***Home Electricity Use*** The rate at which your home consumes electricity is measured in kilowatts. If your home consumes electricity at the rate of 1 kilowatt for 1 hour, you will be charged

26. (b) The answer in (a) corresponds to the area of midpoint rectangles. Part of each rectangle is above the curve and part is below.

for 1 "kilowatt-hour" of electricity. Suppose that the average consumption rate for a certain home is modeled by the function $C(t) = 3.9 - 2.4 \sin(\pi t/12)$, where $C(t)$ is measured in kilowatts and t is the number of hours past midnight. Find the average daily consumption for this home, measured in kilowatt-hours. 93.6 kilowatt-hours

23. *Population Density* Population density measures the number of people per square mile inhabiting a given living area. Washerton's population density, which decreases as you move away from the city center, can be approximated by the function $10{,}000(2 - r)$ at a distance r miles from the city center.

(a) If the population density approaches zero at the edge of the city, what is the city's radius? 2 miles

(b) A thin ring around the center of the city has thickness Δr and radius r. If you straighten it out, it suggests a rectangular strip. Approximately what is its area? $2\pi r \Delta r$

(c) Writing to Learn Explain why the population of the ring in part (b) is approximately

$$10{,}000(2 - r)(2\pi r)\,\Delta r.$$

Population = Population density × Area

(d) Estimate the total population of Washerton by setting up and evaluating a definite integral. ≈83,776

24. *Oil Flow* Oil flows through a cylindrical pipe of radius 3 inches, but friction from the pipe slows the flow toward the outer edge. The speed at which the oil flows at a distance r inches from the center is $8(10 - r^2)$ inches per second.

(a) In a plane cross section of the pipe, a thin ring with thickness Δr at a distance r inches from the center approximates a rectangular strip when you straighten it out. What is the area of the strip (and hence the approximate area of the ring)? $2\pi r \Delta r$

(b) Explain why we know that oil passes through this ring at approximately $8(10 - r^2)(2\pi r)\,\Delta r$ cubic inches per second.

(c) Set up and evaluate a definite integral that will give the rate (in cubic inches per second) at which oil is flowing through the pipe. 396π in^3/sec or ≈ 1244.07 in^3/sec

25. Group Activity *Bagel Sales* From 1995 to 2005, the Konigsberg Bakery noticed a consistent increase in annual sales of its bagels. The annual sales (in thousands of bagels) are shown below.

Year	Sales (thousands)
1995	5
1996	8.9
1997	16
1998	26.3
1999	39.8
2000	56.5
2001	76.4
2002	99.5
2003	125.8
2004	155.3
2005	188

24. (b) $8(10 - r^2)$ in/sec · $(2\pi r)\Delta r$ in^2 = flow in in^3/sec

(a) What was the total number of bagels sold over the 11-year period? (This is not a calculus question!) 797.5 thousand

(b) Use quadratic regression to model the annual bagel sales (in thousands) as a function $B(x)$, where x is the number of years after 1995. $B(x) = 1.6x^2 + 2.3x + 5.0$

(c) Integrate $B(x)$ over the interval $[0, 11]$ to find total bagel sales for the 11-year period. ≈904.02

(d) Explain graphically why the answer in part (a) is smaller than the answer in part (c). See page 389.

26. Group Activity *(Continuation of Exercise 25)*

(a) Integrate $B(x)$ over the interval $[-0.5, 10.5]$ to find total bagel sales for the 11-year period. ≈798.97 thousand

(b) Explain graphically why the answer in part (a) is better than the answer in 25(c).

27. *Filling Milk Cartons* A machine fills milk cartons with milk at an approximately constant rate, but backups along the assembly line cause some variation. The rates (in cases per hour) are recorded at hourly intervals during a 10-hour period, from 8:00 A.M. to 6:00 P.M.

Time	Rate (cases/h)
8	120
9	110
10	115
11	115
12	119
1	120
2	120
3	115
4	112
5	110
6	121

Use the Trapezoidal Rule with $n = 10$ to determine approximately how many cases of milk were filled by the machine over the 10-hour period. 1156.5

28. Writing to Learn As a school project, Anna accompanies her mother on a trip to the grocery store and keeps a log of the car's speed at 10-second intervals. Explain how she can use the data to estimate the distance from her home to the store. What is the connection between this process and the definite integral? See page 389.

29. *Hooke's Law* A certain spring requires a force of 6 N to stretch it 3 cm beyond its natural length.

(a) What force would be required to stretch the string 9 cm beyond its natural length? 18 N

(b) What would be the work done in stretching the string 9 cm beyond its natural length? 81 N · cm

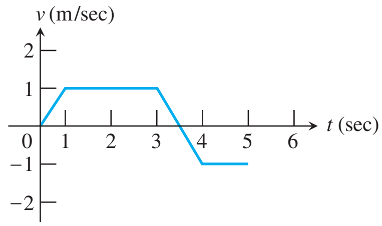
30. *Hooke's Law* Hooke's Law also applies to *compressing* springs; that is, it requires a force of kx to compress a spring a distance x from its natural length. Suppose a 10,000-lb force compressed a spring from its natural length of 12 inches to a length of 11 inches. How much work was done in compressing the spring

(a) the first half-inch? 1250 inch-pounds **(b)** the second half-inch? 3750 inch-pounds

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

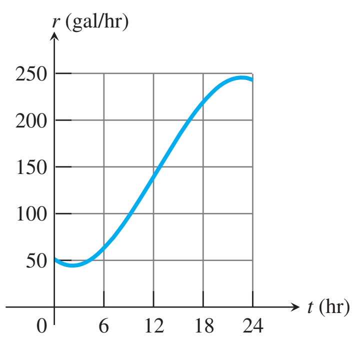
31. True or False The figure below shows the velocity for a particle moving along the x-axis. The displacement for this particle is negative. Justify your answer. False. The displacement is the integral of the velocity from $t = 0$ to $t = 5$ and is positive.



32. True or False If the velocity of a particle moving along the x-axis is always positive, then the displacement is equal to the total distance traveled. Justify your answer.

33. Multiple Choice The graph below shows the rate at which water is pumped from a storage tank. Approximate the total gallons of water pumped from the tank in 24 hours. C

(A) 600 **(B)** 2400 **(C)** 3600 **(D)** 4200 **(E)** 4800



34. Multiple Choice The data for the acceleration $a(t)$ of a car from 0 to 15 seconds are given in the table below. If the velocity at $t = 0$ is 5 ft/sec, which of the following gives the approximate velocity at $t = 15$ using the Trapezoidal Rule? D

(A) 47 ft/sec **(B)** 52 ft/sec **(C)** 120 ft/sec

(D) 125 ft/sec **(E)** 141 ft/sec

t (sec)	0	3	6	9	12	15
$a(t)$ (ft/sec^2)	4	8	6	9	10	10

35. Multiple Choice The rate at which customers arrive at a counter to be served is modeled by the function F defined by $F(t) = 12 + 6\cos\left(\frac{t}{\pi}\right)$ for $0 \le t \le 60$, where $F(t)$ is measured in customers per minute and t is measured in minutes. To the nearest whole number, how many customers arrive at the counter over the 60-minute period? B

(A) 720 **(B)** 725 **(C)** 732 **(D)** 744 **(E)** 756

36. Multiple Choice Pollution is being removed from a lake at a rate modeled by the function $y = 20e^{-0.5t}$ tons/yr, where t is the number of years since 1995. Estimate the amount of pollution removed from the lake between 1995 and 2005. Round your answer to the nearest ton. A

(A) 40 **(B)** 47 **(C)** 56 **(D)** 61 **(E)** 71

Extending the Ideas

37. *Inflation* Although the economy is continuously changing, we analyze it with discrete measurements. The following table records the *annual* inflation rate as measured each month for 13 consecutive months. Use the Trapezoidal Rule with $n = 12$ to find the overall inflation rate for the year. 0.04875

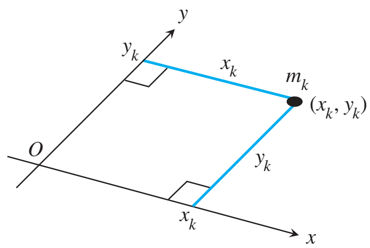
Month	Annual Rate
January	0.04
February	0.04
March	0.05
April	0.06
May	0.05
June	0.04
July	0.04
August	0.05
September	0.04
October	0.06
November	0.06
December	0.05
January	0.05

38. *Inflation Rate* The table below shows the *monthly* inflation rate (in *thousandths*) for energy prices for thirteen consecutive months. Use the Trapezoidal Rule with $n = 12$ to approximate the *annual* inflation rate for the 12-month period running from the middle of the first month to the middle of the last month. 40 thousandths or 0.040

Month	Monthly Rate (in thousandths)
January	3.6
February	4.0
March	3.1
April	2.8
May	2.8
June	3.2
July	3.3
August	3.1
September	3.2
October	3.4
November	3.4
December	3.9
January	4.0

32. True. Since the velocity is positive, the integral of the velocity is equal to the integral of its absolute value, which is the total distance traveled.

39. *Center of Mass* Suppose we have a finite collection of masses in the coordinate plane, the mass m_k located at the point (x_k, y_k) as shown in the figure.



Each mass m_k has **moment $m_k y_k$ with respect to the x-axis** and **moment $m_k x_k$ about the y-axis.** The moments of the entire system with respect to the two axes are

$$\text{Moment about } x\text{-axis: } M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis: } M_y = \sum m_k x_k.$$

The **center of mass** is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}.$$

Suppose we have a thin, flat plate occupying a region in the plane.

(a) Imagine the region cut into thin strips parallel to the y-axis. Show that

$$\bar{x} = \frac{\int x\, dm}{\int dm},$$

where $dm = \delta\, dA$, $\delta =$ density (mass per unit area), and $A =$ area of the region.

(b) Imagine the region cut into thin strips parallel to the x-axis. Show that

$$\bar{y} = \frac{\int y\, dm}{\int dm},$$

where $dm = \delta\, dA$, $\delta =$ density, and $A =$ area of the region.

In Exercises 40 and 41, use Exercise 39 to find the center of mass of the region with given density.

40. the region bounded by the parabola $y = x^2$ and the line $y = 4$ with constant density δ $\bar{x} = 0, \bar{y} = 12/5$

41. the region bounded by the lines $y = x$, $y = -x$, $x = 2$ with constant density δ $\bar{x} = 4/3, \bar{y} = 0$

1. (a) Right: $0 \le t < \pi/2$, $3\pi/2 < t \le 2\pi$
Left: $\pi/2 < t < 3\pi/2$
Stopped: $t = \pi/2, 3\pi/2$
(b) 0; 3 **(c)** 20

2. (a) Right: $0 < t < \pi/3$
Left: $\pi/3 < t \le \pi/2$
Stopped: $t = 0, \pi/3$
(b) 2; 5 **(c)** 6

3. (a) Right: $0 \le t < 5$
Left: $5 < t \le 10$
Stopped: $t = 5$
(b) 0; 3 **(c)** 245

4. (a) Right: $0 \le t < 1$
Left: $1 < t < 2$
Stopped: $t = 1, 2$
(b) 4; 7 **(c)** 6

5. (a) Right: $0 < t < \pi/2$, $3\pi/2 < t < 2\pi$
Left: $\pi/2 < t < \pi$, $\pi < t < 3\pi/2$
Stopped: $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$
(b) 0 ; 3 **(c)** 20/3

6. (a) Right: $0 \le t < 4$
Left: never
Stopped: $t = 4$
(b) 16/3; 25/3 **(c)** 16/3

7. (a) Right: $0 \le t < \pi/2$, $3\pi/2 < t \le 2\pi$
Left: $\pi/2 < t < 3\pi/2$
Stopped: $t = \pi/2, 3\pi/2$
(b) 0; 3 **(c)** $2e - (2/e) \approx 4.7$

8. (a) Right: $0 < t \le 3$
Left: never
Stopped: $t = 0$
(b) $(\ln 10)/2 \approx 1.15$; 4.15 **(c)** $(\ln 10)/2 \approx 1.15$

25. (d) The answer in (a) corresponds to the area of left hand rectangles. These rectangles lie under the curve $B(x)$. The answer in (c) corresponds to the area under the curve. This area is greater than the area of the rectangles.

28. One possible answer:
Plot the speeds vs. time. Connect the points and find the area under the line graph. The definite integral also gives the area under the curve.

39. (a, b) Take $dm = \delta\, dA$ as m_k and letting $dA \to 0$, $k \to \infty$ in the center of mass equations.

7.2 Areas in the Plane

What you'll learn about

- Area Between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with Changing Functions
- Integrating with Respect to y
- Saving Time with Geometric Formulas

. . . and why

The techniques of this section allow us to compute areas of complex regions of the plane.

Area Between Curves

We know how to find the area of a region between a curve and the x-axis but many times we want to know the area of a region that is bounded above by one curve, $y = f(x)$, and below by another, $y = g(x)$ (Figure 7.3).

We find the area as an integral by applying the first two steps of the modeling strategy developed in Section 7.1.

1. We partition the region into vertical strips of equal width Δx and approximate each strip with a rectangle with base parallel to $[a, b]$ (Figure 7.4). Each rectangle has area

$$[f(c_k) - g(c_k)]\,\Delta x$$

for some c_k in its respective subinterval (Figure 7.5). This expression will be nonnegative even if the region lies below the x-axis. We approximate the area of the region with the Riemann sum

$$\sum [f(c_k) - g(c_k)]\,\Delta x.$$

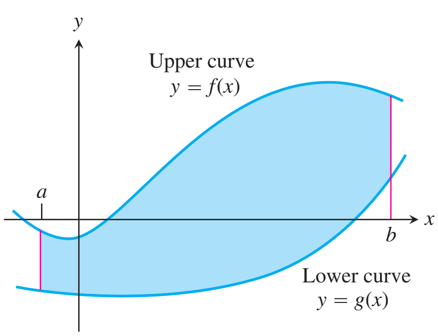


Figure 7.3 The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

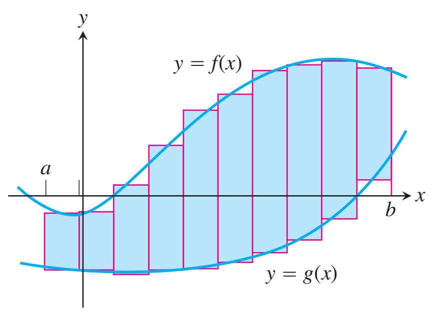


Figure 7.4 We approximate the region with rectangles perpendicular to the x-axis.

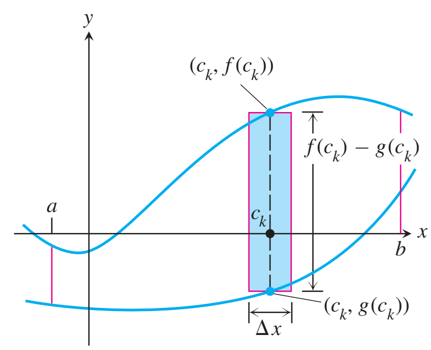


Figure 7.5 The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

2. The limit of these sums as $\Delta x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

This approach to finding area captures the properties of area, so it can serve as a definition.

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $[f - g]$ from a to b,

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

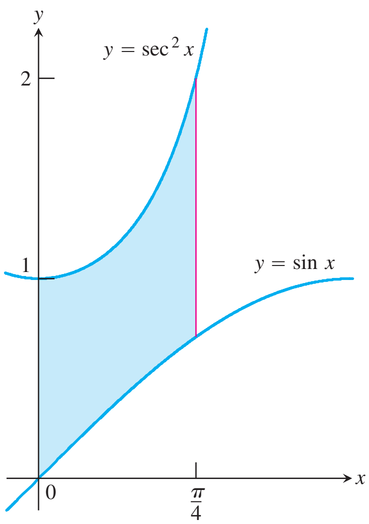


Figure 7.6 The region in Example 1.

EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

SOLUTION

We graph the curves (Figure 7.6) to find their relative positions in the plane, and see that $y = \sec^2 x$ lies *above* $y = \sin x$ on $[0, \pi/4]$. The area is therefore

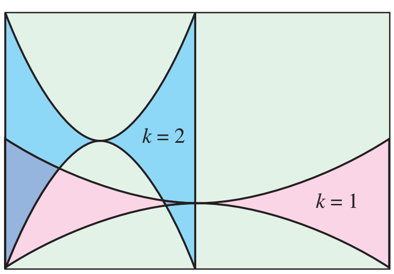
$$A = \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx$$

$$= \Big[\tan x + \cos x\Big]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} \text{ units squared.}$$

Now try Exercise 1.

$y_1 = 2k - k \sin kx$
$y_2 = k \sin kx$



$[0, \pi]$ by $[0, 4]$

Figure 7.7 Two members of the family of butterfly-shaped regions described in Exploration 1.

EXPLORATION 1 A Family of Butterflies

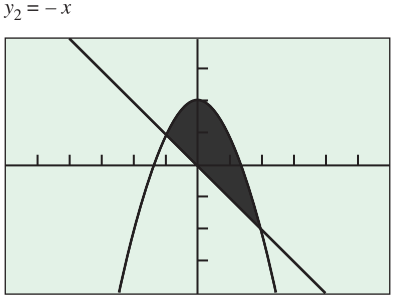
For each positive integer k, let A_k denote the area of the butterfly-shaped region enclosed between the graphs of $y = k \sin kx$ and $y = 2k - k \sin kx$ on the interval $[0, \pi/k]$. The regions for $k = 1$ and $k = 2$ are shown in Figure 7.7.

1. Find the areas of the two regions in Figure 7.7.
2. Make a conjecture about the areas A_k for $k \geq 3$.
3. Set up a definite integral that gives the area A_k. Can you make a simple u-substitution that will transform this integral into the definite integral that gives the area A_1?
4. What is $\lim_{k\to\infty} A_k$?
5. If P_k denotes the perimeter of the kth butterfly-shaped region, what is $\lim_{k\to\infty} P_k$? (You can answer this question without an explicit formula for P_k.)

Area Enclosed by Intersecting Curves

When a region is enclosed by intersecting curves, the intersection points give the limits of integration.

$y_1 = 2 - x^2$
$y_2 = -x$



$[-6, 6]$ by $[-4, 4]$

Figure 7.8 The region in Example 2.

EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

SOLUTION

We graph the curves to view the region (Figure 7.8).

The limits of integration are found by solving the equation

$$2 - x^2 = -x$$

either algebraically or by calculator. The solutions are $x = -1$ and $x = 2$.

continued

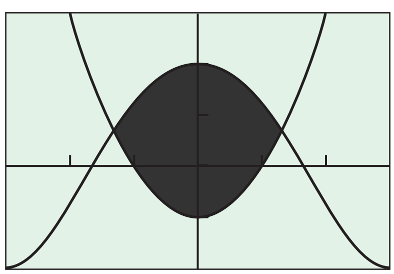
Since the parabola lies above the line on $[-1, 2]$, the area integrand is $2 - x^2 - (-x)$.

$$A = \int_{-1}^{2} [2 - x^2 - (-x)]\,dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \frac{9}{2} \text{ units squared}$$

Now try Exercise 5.



[–3, 3] by [–2, 3]

Figure 7.9 The region in Example 3.

EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

SOLUTION

The region is shown in Figure 7.9.

Using a calculator, we solve the equation

$$2\cos x = x^2 - 1$$

to find the x-coordinates of the points where the curves intersect. These are the limits of integration. The solutions are $x = \pm 1.265423706$. We store the negative value as A and the positive value as B. The area is

$$\text{NINT}\,(2\cos x - (x^2 - 1), x, A, B) \approx 4.994907788.$$

This is the final calculation, so we are now free to round. The area is about 4.99.

Now try Exercise 7.

Boundaries with Changing Functions

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual.

Finding Intersections by Calculator

The coordinates of the points of intersection of two curves are sometimes needed for other calculations. To take advantage of the accuracy provided by calculators, use them to solve for the values and *store* the ones you want.

EXAMPLE 4 Finding Area Using Subregions

Find the area of the region R in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

SOLUTION

The region is shown in Figure 7.10.

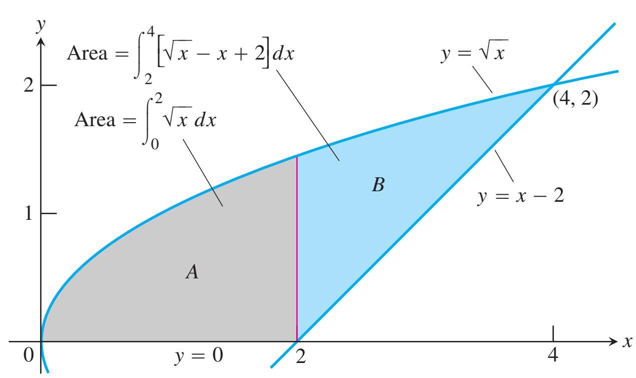


Figure 7.10 Region R split into subregions A and B. (Example 4)

continued

While it appears that no single integral can give the area of R (the bottom boundary is defined by two different curves), we can split the region at $x = 2$ into two regions A and B. The area of R can be found as the sum of the areas of A and B.

$$\text{Area of } R = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 [\sqrt{x} - (x - 2)]\,dx}_{\text{area of } B}$$

$$= \frac{2}{3}x^{3/2}\Big]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

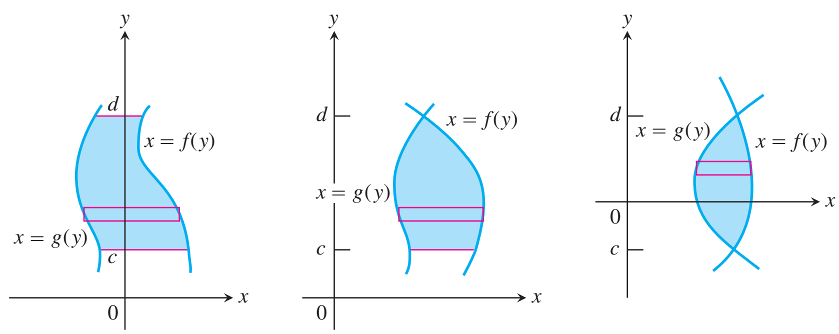
$$= \frac{10}{3} \text{ units squared}$$

Now try Exercise 9.

Integrating with Respect to *y*

Sometimes the boundaries of a region are more easily described by functions of y than by functions of x. We can use approximating rectangles that are horizontal rather than vertical and the resulting basic formula has y in place of x.

For regions like these



use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

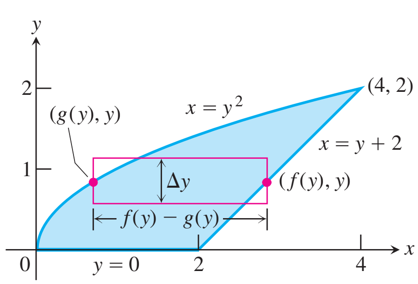


Figure 7.11 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y. (Example 5)

EXAMPLE 5 Integrating with Respect to *y*

Find the area of the region in Example 4 by integrating with respect to y.

SOLUTION

We remarked in solving Example 4 that "it appears that no single integral can give the area of R," but notice how appearances change when we think of our rectangles being summed over y. The interval of integration is $[0, 2]$, and the rectangles run between the same two curves on the entire interval. There is no need to split the region (Figure 7.11).

We need to solve for x in terms of y in both equations:

$$y = x - 2 \quad \text{becomes} \quad x = y + 2,$$

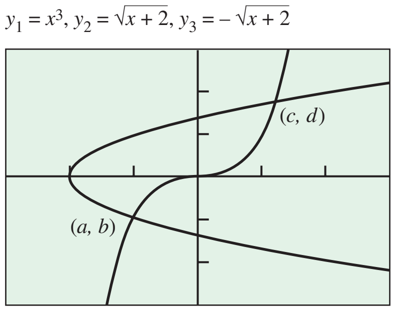
$$y = \sqrt{x} \quad \text{becomes} \quad x = y^2, \quad y \geq 0.$$

continued

We must still be careful to subtract the lower number from the higher number when forming the integrand. In this case, the higher numbers are the higher x-values, which are on the line $x = y + 2$ because the line lies to the *right* of the parabola. So,

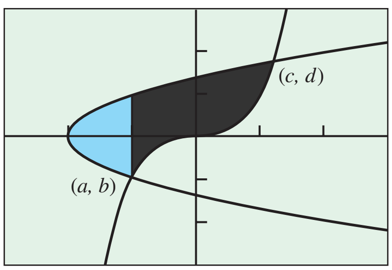
$$\text{Area of } R = \int_0^2 (y + 2 - y^2)\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{10}{3} \text{ units squared.}$$

Now try Exercise 11.



[–3, 3] by [–3, 3]

Figure 7.12 The region in Example 6.



[–3, 3] by [–3, 3]

Figure 7.13 If we integrate with respect to x in Example 6, we must split the region at $x = a$.

EXAMPLE 6 Making the Choice

Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.

SOLUTION

We can produce a graph of the region on a calculator by graphing the three curves $y = x^3$, $y = \sqrt{x+2}$, and $y = -\sqrt{x+2}$ (Figure 7.12).

This conveniently gives us all of our bounding curves as functions of x. If we integrate in terms of x, however, we need to split the region at $x = a$ (Figure 7.13).

On the other hand, we can integrate from $y = b$ to $y = d$ and handle the entire region at once. We solve the cubic for x in terms of y:

$$y = x^3 \quad \text{becomes} \quad x = y^{1/3}.$$

To find the limits of integration, we solve $y^{1/3} = y^2 - 2$. It is easy to see that the lower limit is $b = -1$, but a calculator is needed to find that the upper limit $d = 1.793003715$. We store this value as D.

The cubic lies to the right of the parabola, so

$$\text{Area} = \text{NINT}\,(y^{1/3} - (y^2 - 2), y, -1, \text{D}) = 4.214939673.$$

The area is about 4.21.

Now try Exercise 27.

Saving Time with Geometry Formulas

Here is yet another way to handle Example 4.

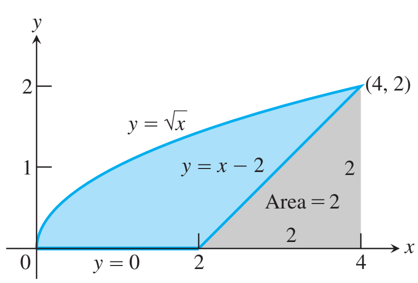


Figure 7.14 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle. (Example 7)

EXAMPLE 7 Using Geometry

Find the area of the region in Example 4 by subtracting the area of the triangular region from the area under the square root curve.

SOLUTION

Figure 7.14 illustrates the strategy, which enables us to integrate with respect to x without splitting the region.

$$\text{Area} = \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) = \frac{2}{3}x^{3/2}\Big]_0^4 - 2 = \frac{10}{3} \text{ units squared}$$

Now try Exercise 35.

The moral behind Examples 4, 5, and 7 is that you often have options for finding the area of a region, some of which may be easier than others. You can integrate with respect to x or with respect to y, you can partition the region into subregions, and sometimes you can even use traditional geometry formulas. Sketch the region first and take a moment to determine the best way to proceed.

Quick Review 7.2 *(For help, go to Sections 1.2 and 5.1.)*

In Exercises 1–5, find the area between the x-axis and the graph of the given function over the given interval.

1. $y = \sin x$ over $[0, \pi]$ 2

2. $y = e^{2x}$ over $[0, 1]$ $\frac{1}{2}(e^2 - 1) \approx 3.195$

3. $y = \sec^2 x$ over $[-\pi/4, \pi/4]$ 2

4. $y = 4x - x^3$ over $[0, 2]$ 4

5. $y = \sqrt{9 - x^2}$ over $[-3, 3]$ $9\pi/2$

In Exercises 6–10, find the x- and y-coordinates of all points where the graphs of the given functions intersect. If the curves never intersect, write "NI."

6. $y = x^2 - 4x$ and $y = x + 6$ (6, 12); (−1, 5)

7. $y = e^x$ and $y = x + 1$ (0, 1)

8. $y = x^2 - \pi x$ and $y = \sin x$ (0, 0); (π, 0)

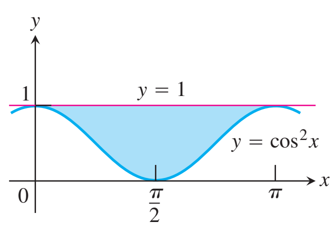
9. $y = \dfrac{2x}{x^2 + 1}$ and $y = x^3$ (−1, −1); (0, 0); (1, 1)

10. $y = \sin x$ and $y = x^3$ (–0.9286, –0.8008); (0, 0); (0.9286, 0.8008)

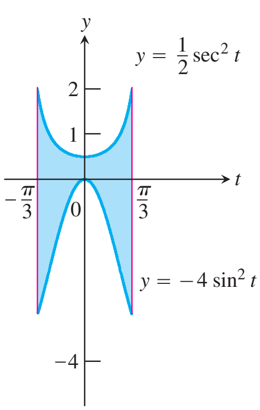
Section 7.2 Exercises

In Exercises 1–6, find the area of the shaded region analytically.

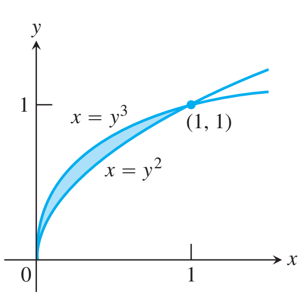
1. $\pi/2$



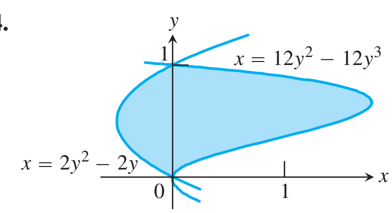
2. $4\pi/3$



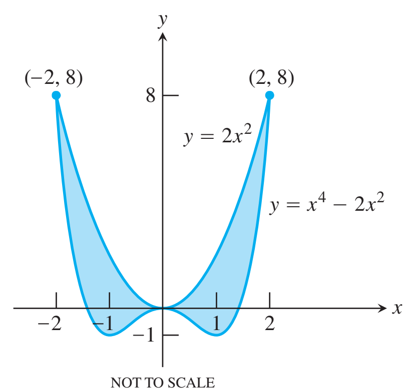
3. 1/12



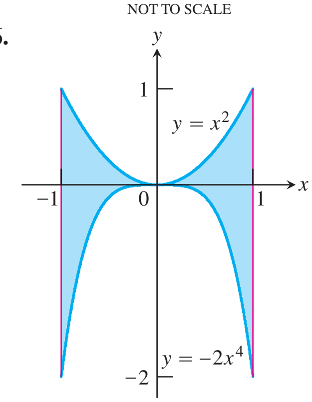
4. 4/3



5. 128/15



6. 22/15

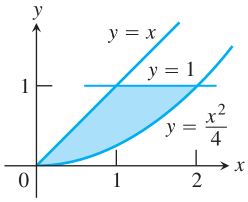


In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

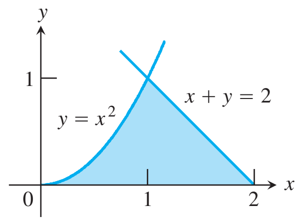
7. $y = \sin x$, $y = 1 - x^2$ ≈ 1.670 **8.** $y = \cos(2x)$, $y = x^2 - 2$ ≈ 4.332

In Exercises 9 and 10, find the area of the shaded region analytically.

9. 5/6



10. 5/6

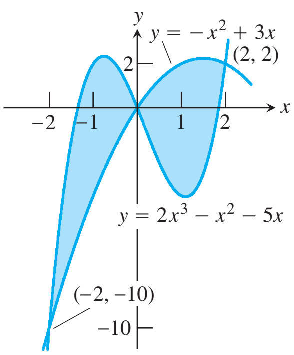


In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by integrating with respect to y.

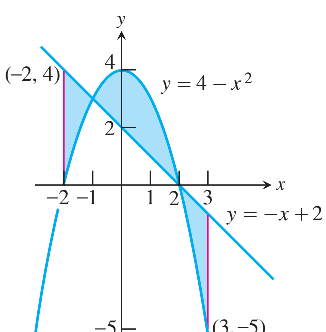
11. $y^2 = x + 1$, $y^2 = 3 - x$ ≈ 7.542 **12.** $y^2 = x + 3$, $y = 2x$ ≈ 7.146

In Exercises 13 and 14, find the total shaded area.

13. 16



14. $8\frac{1}{6}$



In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

15. $y = x^2 - 2$ and $y = 2$ $10\frac{2}{3}$

16. $y = 2x - x^2$ and $y = -3$ $10\frac{2}{3}$

17. $y = 7 - 2x^2$ and $y = x^2 + 4$ 4

18. $y = x^4 - 4x^2 + 4$ and $y = x^2$ 8

19. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$ $\frac{2}{3}a^3$

20. $y = \sqrt{|x|}$ and $5y = x + 6$ $1\frac{2}{3}$ (3 points of intersection)
(How many intersection points are there?)

21. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$ $21\frac{1}{3}$

22. $x = y^2$ and $x = y + 2$ $4\frac{1}{2}$

23. $y^2 - 4x = 4$ and $4x - y = 16$ $30\frac{3}{8}$

24. $x - y^2 = 0$ and $x + 2y^2 = 3$ 4

25. $x + y^2 = 0$ and $x + 3y^2 = 2$ $8/3$

26. $4x^2 + y = 4$ and $x^4 - y = 1$ $6\frac{14}{15}$

27. $x + y^2 = 3$ and $4x + y^2 = 0$ 8

28. $y = 2\sin x$ and $y = \sin 2x$, $0 \le x \le \pi$ 4

29. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \le x \le \pi/3$ $6\sqrt{3}$

30. $y = \cos(\pi x/2)$ and $y = 1 - x^2$ $\frac{4}{3} - \frac{4}{\pi} \approx 0.0601$

31. $y = \sin(\pi x/2)$ and $y = x$ $\frac{4 - \pi}{\pi} \approx 0.273$

32. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, $x = \pi/4$ $\frac{\pi}{2}$

33. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \le y \le \pi/4$ $4 - \pi \approx 0.858$

34. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \le y \le \pi/2$ 2

In Exercises 35 and 36, find the area of the region by subtracting the area of a triangular region from the area of a larger region.

35. The region on or above the x-axis bounded by the curves $y^2 = x + 3$ and $y = 2x$. ≈ 4.333

36. The region on or above the x-axis bounded by the curves $y = 4 - x^2$ and $y = 3x$. 15/2

37. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$. 1/2

38. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis. 1

39. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$. $\sqrt{2} - 1 \approx 0.414$

40. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **(a)** x, **(b)** y. 32/3

41. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

(a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure. $(-\sqrt{c}, c)$; $(\sqrt{c}, c)$

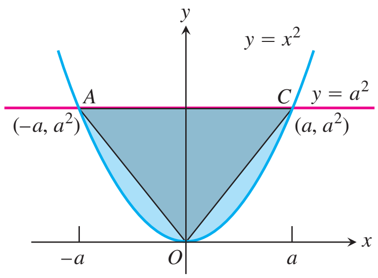
(b) Find c by integrating with respect to y. (This puts c in the limits of integration.) $\int_0^c \sqrt{y}\,dy = \int_c^4 \sqrt{y}\,dy \Rightarrow c = 2^{4/3}$

(c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

42. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$. 11/3

41. (c) $\int_0^{\sqrt{c}} (c - x^2)\,dx = (4 - c)\sqrt{c} + \int_{\sqrt{c}}^{2} (4 - x^2)\,dx \Rightarrow c = 2^{4/3}$

43. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero. 3/4

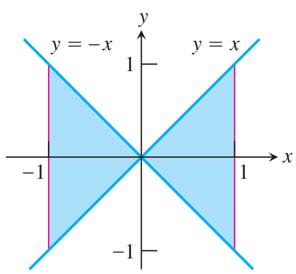


44. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$. 4

45. Writing to Learn Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer. Neither; both are zero

i. $\displaystyle\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

ii. $\displaystyle\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



46. Writing to Learn Is the following statement true, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer. Sometimes; If $f(x) \geq g(x)$ on (a, b), then true.

47. Find the area of the propeller-shaped region enclosed between the graphs of $\ln 4 - (1/2) \approx 0.886$

$$y = \frac{2x}{x^2 + 1} \quad \text{and} \quad y = x^3.$$

48. Find the area of the propeller-shaped region enclosed between the graphs of $y = \sin x$ and $y = x^3$. ≈ 0.4303

49. Find the positive value of k such that the area of the region enclosed between the graph of $y = k\cos x$ and the graph of $y = kx^2$ is 2. $k \approx 1.8269$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

50. True or False The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 10$ is 36. Justify your answer.
True. 36 is the value of the appropriate integral.

51. True or False The area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis is given by the definite integral $\int_0^{0.739} (x - \cos x)\,dx$. Justify your answer. False. It is $\displaystyle\int_0^{0.739} (\cos x - x)\,dx$.

52. Multiple Choice Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = y^2 + 2$, and the line $x = 4$. Which of the following integrals gives the area of R? A

(A) $\displaystyle\int_0^{\sqrt{2}} [4 - (y^2 + 2)]\,dy$ **(B)** $\displaystyle\int_0^{\sqrt{2}} [(y^2 + 2) - 4]\,dy$

(C) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} [4 - (y^2 + 2)]\,dy$ **(D)** $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 + 2) - 4]\,dy$

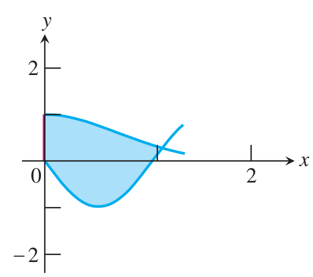
(E) $\displaystyle\int_2^4 [4 - (y^2 + 2)]\,dy$

53. Multiple Choice Which of the following gives the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 3$? E

(A) 2 **(B)** 9/2 **(C)** 13/2 **(D)** 13 **(E)** 27/2

54. Multiple Choice Let R be the shaded region enclosed by the graphs of $y = e^{-x^2}$, $y = -\sin(3x)$, and the y-axis as shown in the figure below. Which of the following gives the approximate area of the region R? B

(A) 1.139 **(B)** 1.445 **(C)** 1.869 **(D)** 2.114 **(E)** 2.340



55. Multiple Choice Let f and g be the functions given by $f(x) = e^x$ and $g(x) = 1/x$. Which of the following gives the area of the region enclosed by the graphs of f and g between $x = 1$ and $x = 2$? A

(A) $e^2 - e - \ln 2$

(B) $\ln 2 - e^2 + e$

(C) $e^2 - \dfrac{1}{2}$

(D) $e^2 - e - \dfrac{1}{2}$

(E) $\dfrac{1}{e} - \ln 2$

Exploration

56. Group Activity *Area of Ellipse*

An ellipse with major axis of length $2a$ and minor axis of length $2b$ can be coordinatized with its center at the origin and its major axis horizontal, in which case it is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Find the equations that define the upper and lower semiellipses as functions of x.

(b) Write an integral expression that gives the area of the ellipse.

(c) With your group, use NINT to find the areas of ellipses for various lengths of a and b.

(d) There is a simple formula for the area of an ellipse with major axis of length $2a$ and minor axis of length $2b$. Can you tell what it is from the areas you and your group have found?

(e) Work with your group to write a *proof* of this area formula by showing that it is the exact value of the integral expression in part (b).

56. (a) $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

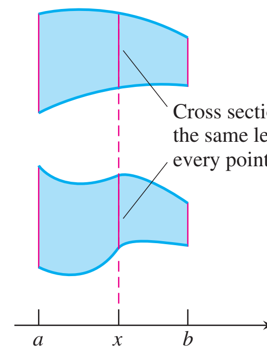
(b) $2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$

(c) Answers may vary.

(d, e) $ab\pi$

Extending the Ideas

57. *Cavalieri's Theorem* Bonaventura Cavalieri (1598–1647) discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point (see figure), then the regions have the same area. Show that this theorem is true.



58. Find the area of the region enclosed by the curves

$$y = \frac{x}{x^2 + 1} \quad \text{and} \quad y = mx, \quad 0 < m < 1.$$

$m - \ln(m) - 1$

57. Since $f(x) - g(x)$ is the same for each region where $f(x)$ and $g(x)$ represent the upper and lower edges, area $= \int_a^b [f(x) - g(x)]\,dx$ will be the same for each.

7.3 Volumes

What you'll learn about

- Volume As an Integral
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

. . . and why

The techniques of this section allow us to compute volumes of certain solids in three dimensions.

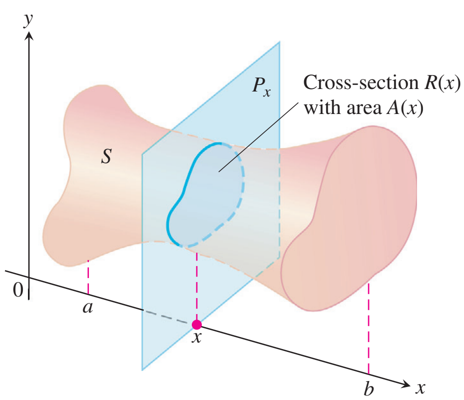


Figure 7.15 The cross section of an arbitrary solid at point x.

Volume As an Integral

In Section 5.1, Example 3, we estimated the volume of a sphere by partitioning it into thin slices that were nearly cylindrical and summing the cylinders' volumes using MRAM. MRAM sums are Riemann sums, and had we known how at the time, we could have continued on to express the volume of the sphere as a definite integral.

Starting the same way, we can now find the volumes of a great many solids by integration. Suppose we want to find the volume of a solid like the one in Figure 7.15. The cross section of the solid at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$. If A is a continuous function of x, we can use it to define and calculate the volume of the solid as an integral in the following way.

We partition $[a, b]$ into subintervals of length Δx and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points. The kth slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between the two planes based on the region $R(x_k)$ (Figure 7.16).

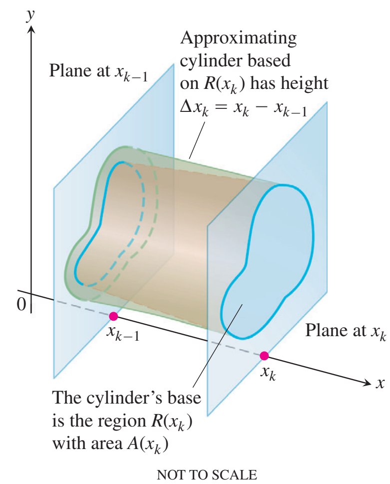


Figure 7.16 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k.

The volume of the cylinder is

$$V_k = \text{base area} \times \text{height} = A(x_k) \times \Delta x.$$

The sum

$$\sum V_k = \sum A(x_k) \times \Delta x$$

approximates the volume of the solid.

This is a Riemann sum for $A(x)$ on $[a, b]$. We expect the approximations to improve as the norms of the partitions go to zero, so we define their limiting integral to be the *volume of the solid.*

DEFINITION Volume of a Solid

The **volume of a solid** of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

To apply the formula in the previous definition, we proceed as follows.

How to Find Volume by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

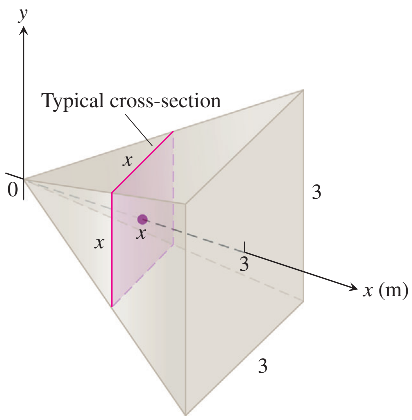


Figure 7.17 A cross section of the pyramid in Example 1.

Square Cross Sections

Let us apply the volume formula to a solid with square cross sections.

EXAMPLE 1 A Square-Based Pyramid

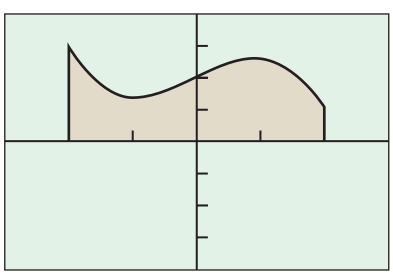
A pyramid 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

SOLUTION

We follow the steps for the method of slicing.

1. *Sketch.* We draw the pyramid with its vertex at the origin and its altitude along the interval $0 \le x \le 3$. We sketch a typical cross section at a point x between 0 and 3 (Figure 7.17).
2. *Find a formula for* $A(x)$. The cross section at x is a square x meters on a side, so
$$A(x) = x^2.$$
3. *Find the limits of integration.* The squares go from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*
$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2 = \frac{x^3}{3}\bigg]_0^3 = 9 \text{ m}^3$$

Now try Exercise 3.



[–3, 3] by [–4, 4]

Figure 7.18 The region in Example 2.

Circular Cross Sections

The only thing that changes when the cross sections of a solid are circular is the formula for $A(x)$. Many such solids are **solids of revolution,** as in the next example.

EXAMPLE 2 A Solid of Revolution

The region between the graph of $f(x) = 2 + x \cos x$ and the x-axis over the interval $[-2, 2]$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

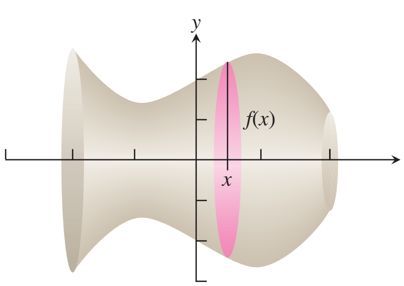


Figure 7.19 The region in Figure 7.18 is revolved about the x-axis to generate a solid. A typical cross section is circular, with radius $f(x) = 2 + x \cos x$. (Example 2)

SOLUTION

Revolving the region (Figure 7.18) about the x-axis generates the vase-shaped solid in Figure 7.19. The cross section at a typical point x is circular, with radius equal to $f(x)$. Its area is
$$A(x) = \pi (f(x))^2.$$

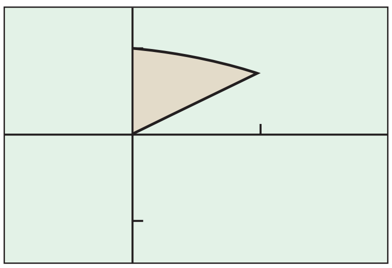
continued

The volume of the solid is

$$V = \int_{-2}^{2} A(x)\,dx$$

$$\approx \text{NINT}\,(\pi(2 + x\cos x)^2, x, -2, 2) \approx 52.43 \text{ units cubed.}$$

Now try Exercise 7.



[–π/4, π/2] by [–1.5, 1.5]

Figure 7.20 The region in Example 3.

EXAMPLE 3 Washer Cross Sections

The region in the first quadrant enclosed by the y-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the x-axis to form a solid. Find its volume.

SOLUTION

The region is shown in Figure 7.20.

We revolve it about the x-axis to generate a solid with a cone-shaped cavity in its center (Figure 7.21).

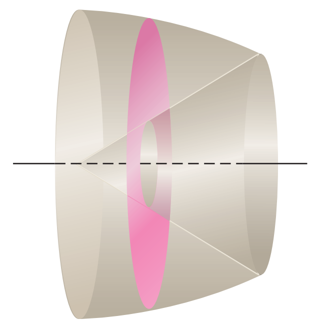


Figure 7.21 The solid generated by revolving the region in Figure 7.20 about the x-axis. A typical cross section is a washer: a circular region with a circular region cut out of its center. (Example 3)

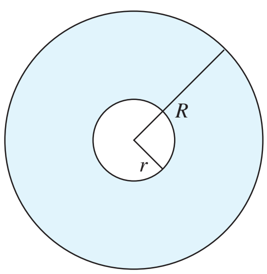


Figure 7.22 The area of a washer is $\pi R^2 - \pi r^2$. (Example 3)

This time each cross section perpendicular to the *axis of revolution* is a *washer,* a circular region with a circular region cut from its center. The area of a washer can be found by subtracting the inner area from the outer area (Figure 7.22).

In our region the cosine curve defines the outer radius, and the curves intersect at $x = \pi/4$. The volume is

$$V = \int_{0}^{\pi/4} \pi(\cos^2 x - \sin^2 x)\,dx$$

$$= \pi\int_{0}^{\pi/4} \cos 2x\,dx \qquad \text{identity: } \cos^2 x - \sin^2 x = \cos 2x$$

$$= \pi\left[\frac{\sin 2x}{2}\right]_{0}^{\pi/4} = \frac{\pi}{2} \text{ units cubed.}$$

Now try Exercise 17.

CAUTION!

The area of a washer is $\pi R^2 - \pi r^2$, which you can simplify to $\pi(R^2 - r^2)$, but *not* to $\pi(R - r)^2$. No matter how tempting it is to make the latter simplification, it's wrong. Don't do it.

We could have done the integration in Example 3 with NINT, but we wanted to demonstrate how a trigonometric identity can be useful under unexpected circumstances in calculus. The double-angle identity turned a difficult integrand into an easy one and enabled us to get an exact answer by antidifferentiation.

Cylindrical Shells

There is another way to find volumes of solids of rotation that can be useful when the axis of revolution is perpendicular to the axis containing the natural interval of integration. Instead of summing volumes of thin slices, we sum volumes of thin cylindrical shells that grow outward from the axis of revolution like tree rings.

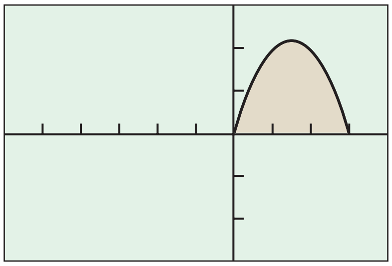


Figure 7.23 The graph of the region in Exploration 1, before revolution.

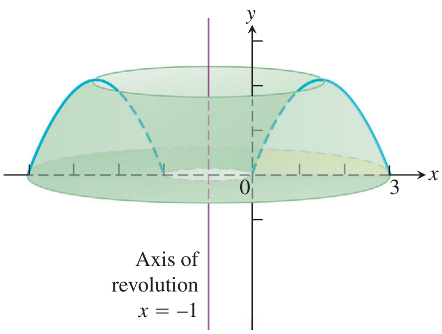


Figure 7.24 The region in Figure 7.23 is revolved about the line $x = -1$ to form a solid cake. The natural interval of integration is along the x-axis, perpendicular to the axis of revolution. (Exploration 1)

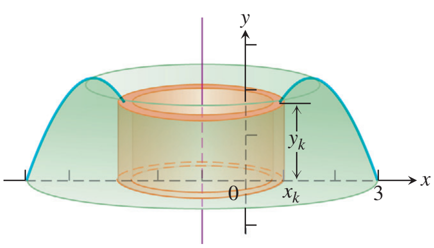


Figure 7.25 Cutting the cake into thin cylindrical slices, working from the inside out. Each slice occurs at some x_k between 0 and 3 and has thickness Δx. (Exploration 1)

EXPLORATION 1 Volume by Cylindrical Shells

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the line $x = -1$ to generate the shape of a cake (Figures 7.23, 7.24). (Such a cake is often called a bundt cake.) What is the volume of the cake?

Integrating with respect to y would be awkward here, as it is not easy to get the original parabola in terms of y. (Try finding the volume by washers and you will soon see what we mean.) To integrate with respect to x, you can do the problem by *cylindrical shells,* which requires that you cut the cake in a rather unusual way.

1. Instead of cutting the usual wedge shape, cut a *cylindrical* slice by cutting straight down all the way around close to the inside hole. Then cut another cylindrical slice around the enlarged hole, then another, and so on. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: smaller to larger, then back to smaller (Figure 7.25). Each slice is sitting over a subinterval of the x-axis of length Δx. Its radius is approximately $(1 + x_k)$. What is its height?

2. If you unroll the cylinder at x_k and flatten it out, it becomes (essentially) a rectangular slab with thickness Δx. Show that the volume of the slab is approximately $2\pi(x_k + 1)(3x_k - x_k{}^2)\Delta x$.

3. $\sum 2\pi(x_k + 1)(3x_k - x_k{}^2)\Delta x$ is a Riemann sum. What is the limit of these Riemann sums as $\Delta x \to 0$?

4. Evaluate the integral you found in step 3 to find the volume of the cake!

EXAMPLE 4 Finding Volumes Using Cylindrical Shells

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

SOLUTION

1. Sketch the region and draw a line segment across it parallel to the axis of revolution (Figure 7.26). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. (We drew the shell in Figure 7.27, but you need not do that.)

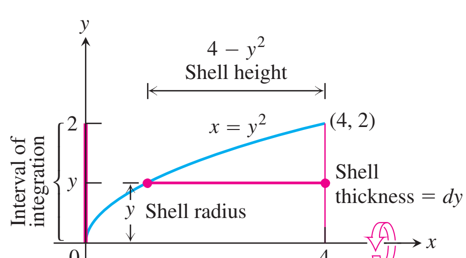


Figure 7.26 The region, shell dimensions, and interval of integration in Example 4.

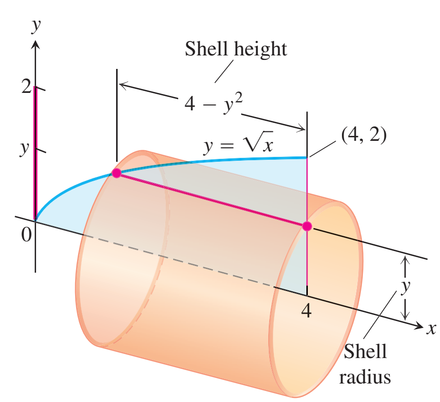


Figure 7.27 The shell swept out by the line segment in Figure 7.26.

2. Identify the limits of integration: y runs from 0 to 2.
3. Integrate to find the volume.

$$V = \int_0^2 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

$$= \int_0^2 2\pi(y)(4 - y^2)\, dy = 8\pi$$

Now try Exercise 33(a).

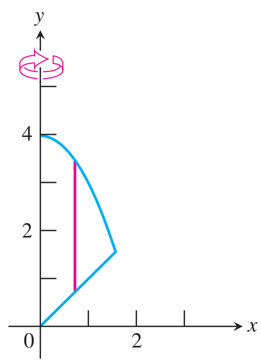


Figure 7.28 The region and the height of a typical shell in Example 5.

EXAMPLE 5 Finding Volumes Using Cylindrical Shells

The region bounded by the curves $y = 4 - x^2$, $y = x$, and $x = 0$ is revolved about the y-axis to form a solid. Use cylindrical shells to find the volume of the solid.

SOLUTION

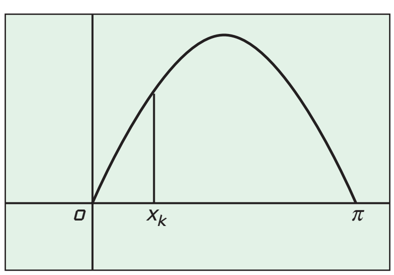
1. Sketch the region and draw a line segment across it parallel to the y-axis (Figure 7.28). The segment's length (shell height) is $4 - x^2 - x$. The distance of the segment from the axis of revolution (shell radius) is x.
2. Identify the limits of integration: The x-coordinate of the point of intersection of the curves $y = 4 - x^2$ and $y = x$ in the first quadrant is about 1.562. So x runs from 0 to 1.562.
3. Integrate to find the volume.

$$V = \int_0^{1.562} 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$= \int_0^{1.562} 2\pi(x)(4 - x^2 - x)\, dx$$

$$\approx 13.327$$

Now try Exercise 35.



[–1, 3.5] by [–0.8, 2.2]

Figure 7.29 The base of the paperweight in Example 6. The segment perpendicular to the x-axis at x_k is the diameter of a semicircle that is perpendicular to the base.

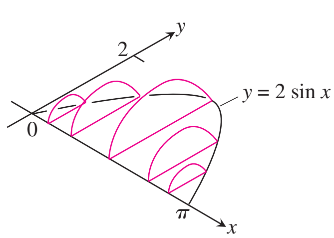


Figure 7.30 Cross sections perpendicular to the region in Figure 7.29 are semicircular. (Example 6)

Other Cross Sections

The method of cross-section slicing can be used to find volumes of a wide variety of unusually shaped solids, so long as the cross sections have areas that we can describe with some formula. Admittedly, it does take a special artistic talent to *draw* some of these solids, but a crude picture is usually enough to suggest how to set up the integral.

EXAMPLE 6 A Mathematician's Paperweight

A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2 \sin x$ (linear units in inches). Each cross section cut perpendicular to the x-axis (and hence to the xy-plane) is a semicircle whose diameter runs from the x-axis to the curve. (Think of the cross section as a semicircular fin sticking up out of the plane.) Find the volume of the paperweight.

SOLUTION

The paperweight is not easily drawn, but we know what it looks like. Its base is the region in Figure 7.29, and the cross sections perpendicular to the base are semicircular fins like those in Figure 7.30.

The semicircle at each point x has

$$\text{radius} = \frac{2 \sin x}{2} = \sin x \quad \text{and area} \quad A(x) = \frac{1}{2}\pi(\sin x)^2.$$

continued

The volume of the paperweight is

$$V = \int_0^{\pi} A(x)\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi} (\sin x)^2\,dx$$

$$\approx \frac{\pi}{2}\,\text{NINT}\,((\sin x)^2, x, 0, \pi)$$

$$\approx \frac{\pi}{2}(1.570796327).$$

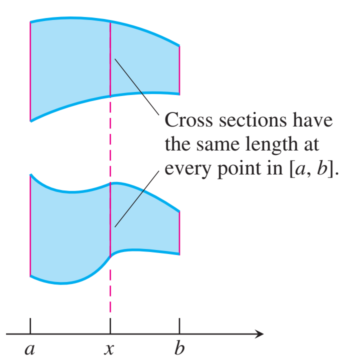
The number in parentheses looks like half of π, an observation that can be confirmed analytically, and which we support numerically by dividing by π to get 0.5. The volume of the paperweight is

$$\frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} \approx 2.47 \text{ in}^3.$$

Now try Exercise 39(a).

Bonaventura Cavalieri
(1598–1647)

Cavalieri, a student of Galileo, discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area. This theorem and a letter of recommendation from Galileo were enough to win Cavalieri a chair at the University of Bologna in 1629. The solid geometry version in Example 7, which Cavalieri never proved, was named after him by later geometers.



EXAMPLE 7 Cavalieri's Volume Theorem

Cavalieri's volume theorem says that solids with equal altitudes and identical cross section areas at each height have the same volume (Figure 7.31). This follows immediately from the definition of volume, because the cross section area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

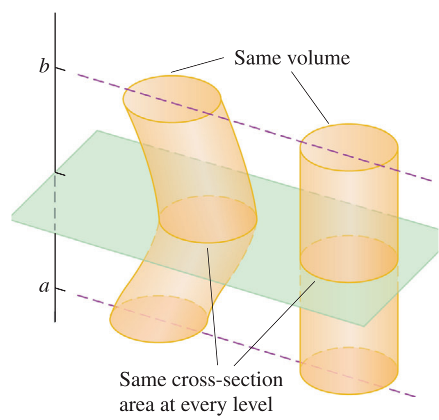
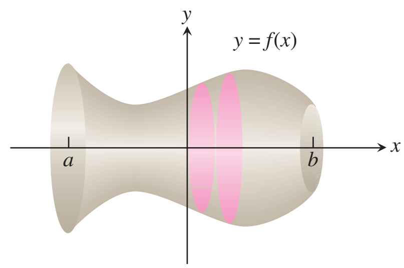


Figure 7.31 *Cavalieri's volume theorem:* These solids have the same volume. You can illustrate this yourself with stacks of coins. (Example 7)

Now try Exercise 43.

EXPLORATION 2 Surface Area

We know how to find the volume of a solid of revolution, but how would we find the *surface area*? As before, we partition the solid into thin slices, but now we wish to form a Riemann sum of approximations to *surface areas of slices* (rather than of volumes of slices).



A typical slice has a surface area that can be approximated by $2\pi \cdot f(x) \cdot \Delta s$, where Δs is the tiny *slant height* of the slice. We will see in Section 7.4, when we study arc length, that $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$, and that this can be written as $\Delta s = \sqrt{1 + (f'(x_k))^2}\, \Delta x$.

Thus, the surface area is approximated by the Riemann sum

$$\sum_{k=1}^{n} 2\pi f(x_k) \sqrt{1 + (f'(x_k))^2}\, \Delta x.$$

1. Write the limit of the Riemann sums as a definite integral from a to b. When will the limit exist?
2. Use the formula from part 1 to find the surface area of the solid generated by revolving a single arch of the curve $y = \sin x$ about the x-axis.
3. The region enclosed by the graphs of $y^2 = x$ and $x = 4$ is revolved about the x-axis to form a solid. Find the surface area of the solid.

Quick Review 7.3 *(For help, go to Section 1.2.)*

In Exercises 1–10, give a formula for the area of the plane region in terms of the single variable x.

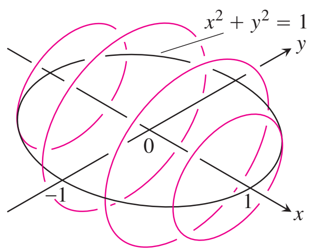
1. a square with sides of length x x^2
2. a square with diagonals of length x $x^2/2$
3. a semicircle of radius x $\pi x^2/2$
4. a semicircle of diameter x $\pi x^2/8$
5. an equilateral triangle with sides of length x $(\sqrt{3}/4)x^2$
6. an isosceles right triangle with legs of length x $x^2/2$
7. an isosceles right triangle with hypotenuse x $x^2/4$
8. an isosceles triangle with two sides of length $2x$ and one side of length x $(\sqrt{15}/4)x^2$
9. a triangle with sides $3x$, $4x$, and $5x$ $6x^2$
10. a regular hexagon with sides of length x $(3\sqrt{3}/2)x^2$

Section 7.3 Exercises

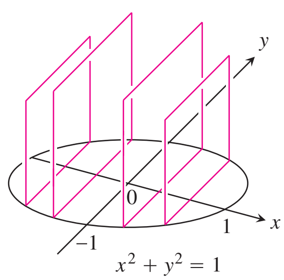
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid that are perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

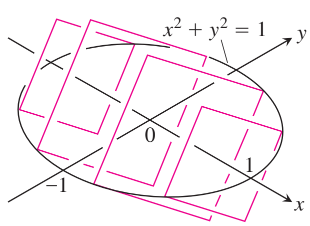
(a) The cross sections are circular disks with diameters in the xy-plane. $\pi(1 - x^2)$



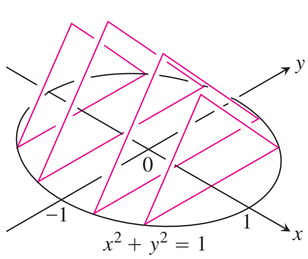
(b) The cross sections are squares with bases in the xy-plane. $4(1 - x^2)$



(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.) $2(1 - x^2)$

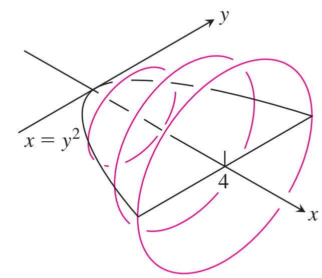


(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}(1 - x^2)$

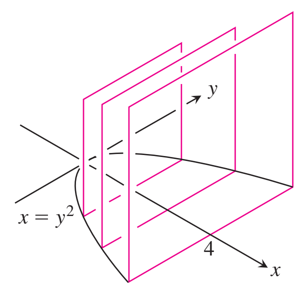


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(a) The cross sections are circular disks with diameters in the xy-plane. πx



(b) The cross sections are squares with bases in the xy-plane. $4x$



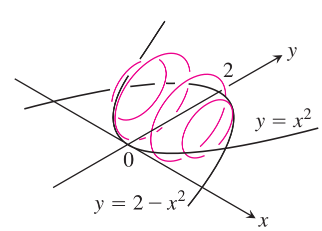
(c) The cross sections are squares with diagonals in the xy-plane. $2x$

(d) The cross sections are equilateral triangles with bases in the xy-plane. $\sqrt{3}x$

In Exercises 3–6, find the volume of the solid analytically.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from $y = -\sqrt{x}$ to $y = \sqrt{x}$. 16

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. $16\pi/15$

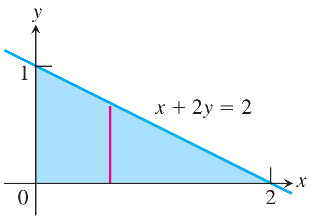


5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. 16/3

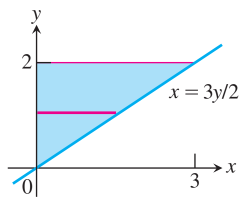
6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. $8/3$

In Exercises 7–10, find the volume of the solid generated by revolving the shaded region about the given axis.

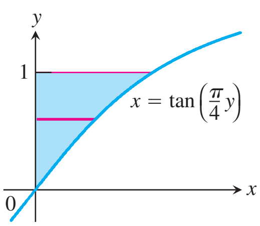
7. about the x-axis $2\pi/3$



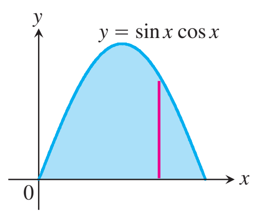
8. about the y-axis 6π



9. about the y-axis $4 - \pi$



10. about the x-axis $\pi^2/16$



In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.

11. $y = x^2, \quad y = 0, \quad x = 2$ $32\pi/5$

12. $y = x^3, \quad y = 0, \quad x = 2$ $128\pi/7$

13. $y = \sqrt{9 - x^2}, \quad y = 0$ 36π

14. $y = x - x^2, \quad y = 0$ $\pi/30$

15. $y = x, \quad y = 1, \quad x = 0$ $2\pi/3$

16. $y = 2x, \quad y = x, \quad x = 1$ π

17. $y = x^2 + 1, \quad y = x + 3$ $117\pi/5$

18. $y = 4 - x^2, \quad y = 2 - x$ $108\pi/5$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$ $\pi^2 - 2\pi$

20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$ 8π

In Exercises 21 and 22, find the volume of the solid generated by revolving the region about the given line.

21. the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$ 2.301

22. the region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$ $\pi(3\pi - 8)$

In Exercises 23–28, find the volume of the solid generated by revolving the region about the y-axis.

23. the region enclosed by $x = \sqrt{5}y^2, \; x = 0, \; y = -1, \; y = 1$ 2π

24. the region enclosed by $x = y^{3/2}, \; x = 0, \; y = 2$ 4π

25. the region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1) $4\pi/3$

26. the region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1) $2\pi/3$

27. the region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$ 8π

28. the region bounded above by the curve $y = \sqrt{x}$ and below by the line $y = x$ $2\pi/15$

Group Activity In Exercises 29–32, find the volume of the solid described.

29. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

(a) the x-axis. 8π **(b)** the y-axis. $32\pi/5$

(c) the line $y = 2$. $8\pi/3$ **(d)** the line $x = 4$. $224\pi/15$

30. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

(a) the line $x = 1$. $2\pi/3$ **(b)** the line $x = 2$. $8\pi/3$

31. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

(a) the line $y = 1$. $16\pi/15$ **(b)** the line $y = 2$. $56\pi/15$

(c) the line $y = -1$. $64\pi/15$

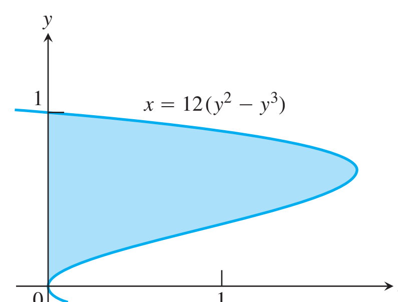
32. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about

(a) the x-axis. $(\pi/3)bh^2$ **(b)** the y-axis. $(\pi/3)b^2h$

In Exercises 33 and 34, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

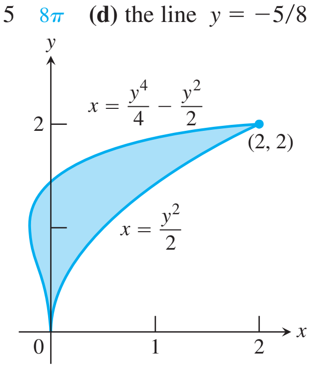
33. (a) the x-axis $6\pi/5$ **(b)** the line $y = 1$ $4\pi/5$

(c) the line $y = 8/5$ 2π **(d)** the line $y = -2/5$ 2π



34. (a) the x-axis $8\pi/3$ **(b)** the line $y = 2$ $8\pi/5$

(c) the line $y = 5$ 8π **(d)** the line $y = -5/8$ 4π



In Exercises 35–38, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

35. $y = x, \quad y = -x/2, \quad x = 2$ 8π

36. $y = x^2, \quad y = 2 - x, \quad x = 0, \quad$ for $x \geq 0$ $5\pi/6$

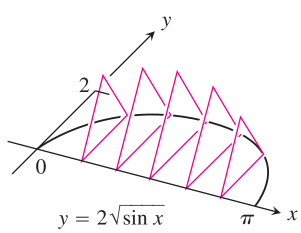
37. $y = \sqrt{x}, \quad y = 0, \quad x = 4$ $128\pi/5$

38. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$ $7\pi/15$

In Exercises 39–42, find the volume of the solid analytically.

39. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross sections perpendicular to the x-axis are

(a) equilateral triangles with bases running from the x-axis to the curve as shown in the figure. $2\sqrt{3}$



(b) squares with bases running from the x-axis to the curve. 8

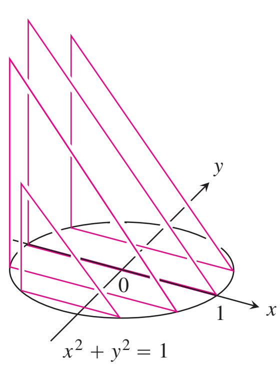
40. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x-axis are

(a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$. $\pi\sqrt{3} - (\pi^2/6)$

(b) squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$. $4\sqrt{3} - (2\pi/3)$

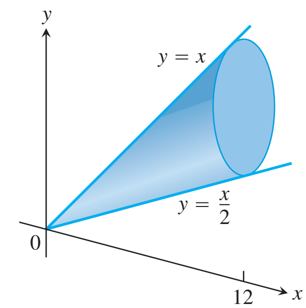
41. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$. 8π

42. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk. $8/3$



43. Writing to Learn A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.

The volumes are equal by Cavalieri's Theorem.



44. *A Twisted Solid* A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross sections.

(a) Find the volume of the column. s^2h

(b) Writing to Learn What will the volume be if the square turns twice instead of once? Give reasons for your answer. s^2h

45. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = x^3$ and $y = 4x$ about

(a) the x-axis, $512\pi/21$

(b) the line $y = 8$. $832\pi/21$

46. Find the volume of the solid generated by revolving the region bounded by $y = 2x - x^2$ and $y = x$ about

(a) the y-axis, $\pi/6$

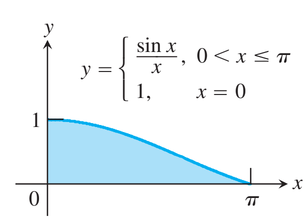
(b) the line $x = 1$. $\pi/6$

47. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by **(a)** the washer method and **(b)** the cylindrical shell method. **(a)** $11\pi/48$ **(b)** $11\pi/48$

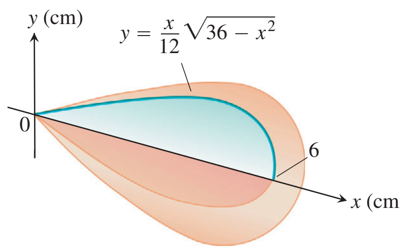
48. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0. \end{cases}$

(a) Show that $x f(x) = \sin x, \quad 0 \leq x \leq \pi$.

(b) Find the volume of the solid generated by revolving the shaded region about the y-axis. 4π



49. ***Designing a Plumb Bob*** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here.



(a) Find the plumb bob's volume. $36\pi/5$ cm^3

(b) If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh to the nearest gram? 192.3 g

50. ***Volume of a Bowl*** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

(a) Find the volume of the bowl. 25π

(b) If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep? $3/(8\pi)$

51. ***The Classical Bead Problem*** A round hole is drilled through the center of a spherical solid of radius r. The resulting cylindrical hole has height 4 cm.

(a) What is the volume of the solid that remains? $32\pi/3$

(b) What is unusual about the answer? The answer is independent of r.

52. **Writing to Learn** Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it. See page 410.

53. ***Same Volume about Each Axis*** The region in the first quadrant enclosed between the graph of $y = ax - x^2$ and the x-axis generates the same volume whether it is revolved about the x-axis or the y-axis. Find the value of a. 5

54. *(Continuation of Exploration 2)* Let $x = g(y) > 0$ have a continuous first derivative on $[c, d]$. Show that the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is See page 410.

$$S = \int_c^d 2\pi\, g(y)\sqrt{1 + (g'(y))^2}\, dy.$$

In Exercises 55–62, find the area of the surface generated by revolving the curve about the indicated axis.

55. $x = \sqrt{y}, \quad 0 \le y \le 2; \quad y\text{-axis}$ ≈ 13.614

56. $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$ ≈ 0.638

57. $x = y^{1/2} - (1/3)^{3/2}, \quad 1 \le y \le 3; \quad y\text{-axis}$ ≈ 16.110

58. $x = \sqrt{2y - 1}, \quad (5/8) \le y \le 1; \quad y\text{-axis}$ ≈ 2.999

59. $y = x^2, \quad 0 \le x \le 2; \quad x\text{-axis}$ ≈ 53.226

60. $y = 3x - x^2, \quad 0 \le x \le 3; \quad x\text{-axis}$ ≈ 44.877

61. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$ ≈ 6.283

62. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x\text{-axis}$ ≈ 51.313

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

63. **True or False** The volume of a solid of a known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is $\int_a^b A(x)\,dx$. Justify your answer. True, by definition.

64. **True or False** If the region enclosed by the y-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the y-axis, the volume of the solid is given by the definite integral $\int_0^2 \pi y^2\,dy$. Justify your answer. False. The volume is given by $\int_0^2 \pi y^4\,dy$.

65. **Multiple Choice** The base of a solid S is the region enclosed by the graph of $y = \ln x$, the line $x = e$, and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, which of the following gives the best approximation of the volume of S? A

(A) 0.718 **(B)** 1.718 **(C)** 2.718 **(D)** 3.171 **(E)** 7.388

66. **Multiple Choice** Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{3/2}$, the x-axis, and the y-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the x-axis? E

(A) 60.3 **(B)** 115.2 **(C)** 225.4 **(D)** 319.7 **(E)** 361.9

67. **Multiple Choice** Let R be the region enclosed by the graph of $y = x^2$, the line $x = 4$, and the x-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the y-axis? B

(A) 64π **(B)** 128π **(C)** 256π **(D)** 360 **(E)** 512

68. **Multiple Choice** Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^x$, and $x = 1$. Which of the following gives the volume of the solid generated when R is revolved about the x-axis? D

(A) $\int_0^1 (e^x - e^{-x})\,dx$

(B) $\int_0^1 (e^{2x} - e^{-2x})\,dx$

(C) $\int_0^1 (e^x - e^{-x})^2\,dx$

(D) $\pi\int_0^1 (e^{2x} - e^{-2x})\,dx$

(E) $\pi\int_0^1 (e^x - e^{-x})^2\,dx$

Explorations

69. *Max-Min* The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the figure.

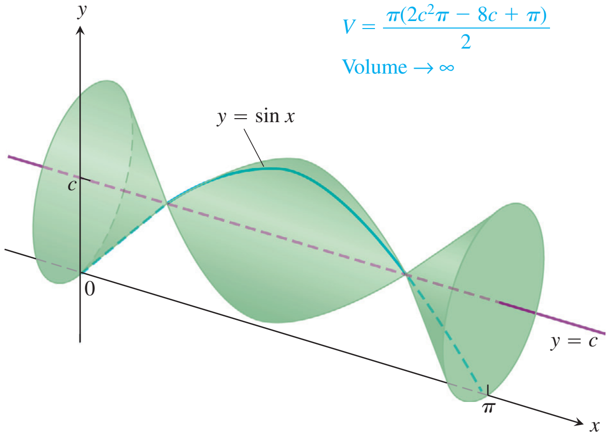
(a) Find the value of c that minimizes the volume of the solid. What is the minimum volume? $\frac{2}{\pi}, \frac{\pi^2 - 8}{2}$

(b) What value of c in $[0, 1]$ maximizes the volume of the solid? 0

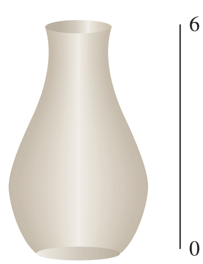
(c) Writing to Learn Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.

$V = \frac{\pi(2c^2\pi - 8c + \pi)}{2}$

Volume $\to \infty$



70. *A Vase* We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 inches. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)



Circumferences	
5.4	10.8
4.5	11.6
4.4	11.6
5.1	10.8
6.3	9.0
7.8	6.3
9.4	

(a) Find the areas of the cross sections that correspond to the given circumferences. 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2

(b) Express the volume of the vase as an integral with respect to y over the interval $[0, 6]$. $\frac{1}{4\pi}\int_0^6 C(y)^2\,dy$

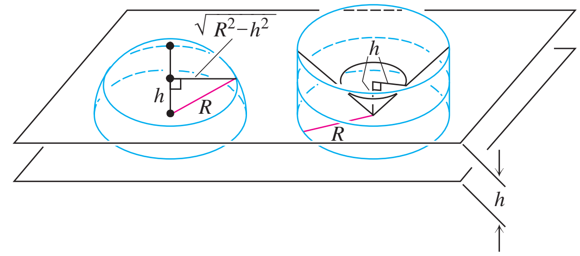
(c) Approximate the integral using the Trapezoidal Rule with $n = 12$. $\approx 34.7 \text{ in}^3$

52. Partition the appropriate interval on the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\,dx$.

54. For a tiny horizontal slice,
slant height $= \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + (g'(y))^2}\,\Delta y$. So the surface area is approximated by the Riemann sum
$\sum_{k=1}^{n} 2\pi\, g(y_k)\sqrt{1 + (g'(y))^2}\,\Delta y.$
The limit of that is the integral.

Extending the Ideas

71. *Volume of a Hemisphere* Derive the formula $V = (2/3)\,\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross sections with the cross sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed as suggested by the figure.



72. *Volume of a Torus* The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut, called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = \pi a^2/2$, since it is the area of a semicircle of radius a.) $2a^2b\pi^2$

73. *Filling a Bowl*

(a) *Volume* A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl. $\pi h^2(3a - h)/3$

(b) *Related Rates* Water runs into a sunken concrete hemispherical bowl of radius 5 m at a rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep? $1/(120\pi)$ m/sec

74. *Consistency of Volume Definitions* The volume formulas in calculus are consistent with the standard formulas from geometry in the sense that they agree on objects to which both apply.

(a) As a case in point, show that if you revolve the region enclosed by the semicircle $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis to generate a solid sphere, the calculus formula for volume at the beginning of the section will give $(4/3)\pi a^3$ for the volume just as it should.

(b) Use calculus to find the volume of a right circular cone of height h and base radius r.

71. Hemisphere cross sectional area:
$\pi(\sqrt{R^2 - h^2})^2 = A_1$
Right circular cylinder with cone removed cross sectional area:
$\pi R^2 - \pi h^2 = A_2$
Since $A_1 = A_2$, the two volumes are equal by Cavalieri's theorem. Thus,
volume of hemisphere = volume of cylinder − volume of cone
$$= \pi R^3 - \tfrac{1}{3}\pi R^3 = \tfrac{2}{3}\pi R^3.$$

74. (a) A cross section has radius $r = \sqrt{a^2 - x^2}$ and area $A(x) = \pi r^2 = \pi(a^2 - x^2)$.
$V = \int_{-a}^{a} \pi(a^2 - x^2) = \frac{4}{3}\pi a^3$

(b) A cross section has radius $x = r\left(1 - \frac{y}{h}\right)$ and
area $A(y) = \pi x^2 = \pi r^2\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right)$.
$V = \int_0^h \pi r^2\left(1 - \frac{2y}{h} + \frac{y^2}{h}\right)dy = \frac{1}{3}\pi r^2 h$

Quick Quiz for AP* Preparation: Sections 7.1–7.3

You may use a graphing calculator to solve the following problems.

1. Multiple Choice The base of a solid is the region in the first quadrant bounded by the x-axis, the graph of $y = \sin^{-1} x$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume? C

(A) 0.117 **(B)** 0.285 **(C)** 0.467 **(D)** 0.571 **(E)** 1.571

2. Multiple Choice Let R be the region in the first quadrant bounded by the graph of $y = 3x - x^2$ and the x-axis. A solid is generated when R is revolved about the vertical line $x = -1$. Set up, but do not evaluate, the definite integral that gives the volume of this solid. A

(A) $\int_0^3 2\pi(x+1)(3x - x^2)\,dx$

(B) $\int_{-1}^3 2\pi(x+1)(3x - x^2)\,dx$

(C) $\int_0^3 2\pi(x)(3x - x^2)\,dx$

(D) $\int_0^3 2\pi(3x - x^2)^2\,dx$

(E) $\int_0^3 (3x - x^2)\,dx$

3. Multiple Choice A developing country consumes oil at a rate given by $r(t) = 20e^{0.2t}$ million barrels per year, where t is time measured in years, for $0 \le t \le 10$. Which of the following expressions gives the amount of oil consumed by the country during the time interval $0 \le t \le 10$? D

(A) $r(10)$

(B) $r(10) - r(0)$

(C) $\int_0^{10} r'(t)\,dt$

(D) $\int_0^{10} r(t)\,dt$

(E) $10 \cdot r(10)$

4. Free Response Let R be the region bounded by the graphs of $y = \sqrt{x}$, $y = e^{-x}$, and the y-axis.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = -1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a semicircle whose diameter runs from the graph of $y = \sqrt{x}$ to the graph of $y = e^{-x}$. Find the volume of this solid.

4. (a) The two graphs intersect where $\sqrt{x} = e^{-x}$, which a calculator shows to be $x = 0.42630275$. Store this value as A.

The area of R is $\int_0^A (e^{-x} - \sqrt{x})\,dx = 0.162$.

(b) Volume $= \int_0^A \pi((e^{-x} + 1)^2 - (\sqrt{x} + 1)^2)\,dx = 1.631$.

(c) Volume $= \int_0^A \frac{1}{2}\pi\left(\frac{e^{-x} - \sqrt{x}}{2}\right)^2 dx = 0.035$.

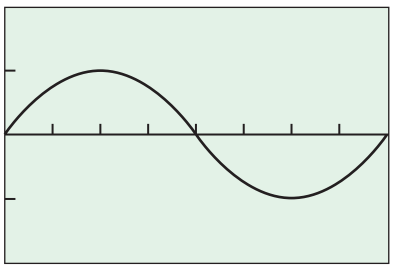
7.4 Lengths of Curves

What you'll learn about

- A Sine Wave
- Length of a Smooth Curve
- Vertical Tangents, Corners, and Cusps

. . . and why

The length of a smooth curve can be found using a definite integral.



[0, 2π] by [–2, 2]

Figure 7.32 One wave of a sine curve has to be longer than 2π.

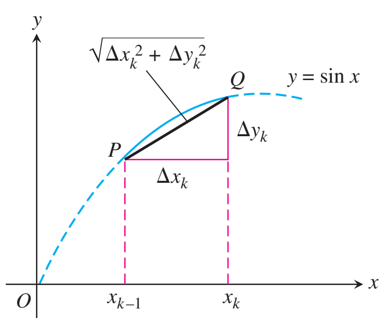


Figure 7.33 The line segment approximating the arc PQ of the sine curve above the subinterval $[x_{k-1}, x_k]$. (Example 1)

Group Exploration

Later in this section we will use an integral to find the length of the sine wave with great precision. But there are ways to get good approximations without integrating. Take five minutes to come up with a written estimate of the curve's length. No fair looking ahead.

A Sine Wave

How long is a sine wave (Figure 7.32)?

The usual meaning of *wavelength* refers to the fundamental period, which for $y = \sin x$ is 2π. But how long is the curve itself? If you straightened it out like a piece of string along the positive x-axis with one end at 0, where would the other end be?

EXAMPLE 1 The Length of a Sine Wave

What is the length of the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$?

SOLUTION

We answer this question with integration, following our usual plan of breaking the whole into measurable parts. We partition $[0, 2\pi]$ into intervals so short that the pieces of curve (call them "arcs") lying directly above the intervals are nearly straight. That way, each arc is nearly the same as the line segment joining its two ends and we can take the length of the segment as an approximation to the length of the arc.

Figure 7.33 shows the segment approximating the arc above the subinterval $[x_{k-1}, x_k]$. The length of the segment is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. The sum

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

over the entire partition approximates the length of the curve. All we need now is to find the limit of this sum as the norms of the partitions go to zero. That's the usual plan, but this time there is a problem. Do you see it?

The problem is that the sums as written are not Riemann sums. They do not have the form $\sum f(c_k)\,\Delta x$. We can rewrite them as Riemann sums if we multiply and divide each square root by Δx_k.

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \frac{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}{\Delta x_k}\,\Delta x_k$$

$$= \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

This is better, but we still need to write the last square root as a function evaluated at some c_k in the kth subinterval. For this, we call on the Mean Value Theorem for differentiable functions (Section 4.2), which says that since $\sin x$ is continuous on $[x_{k-1}, x_k]$ and is differentiable on (x_{k-1}, x_k) there is a point c_k in $[x_{k-1}, x_k]$ at which $\Delta y_k/\Delta x_k = \sin' c_k$ (Figure 7.34). That gives us

$$\sum \sqrt{1 + (\sin' c_k)^2}\,\Delta x_k,$$

which *is* a Riemann sum.

Now we take the limit as the norms of the subdivisions go to zero and find that the length of one wave of the sine function is

$$\int_0^{2\pi} \sqrt{1 + (\sin' x)^2}\,dx = \int_0^{2\pi} \sqrt{1 + \cos^2 x}\,dx \approx 7.64. \quad \text{Using NINT}$$

How close was your estimate? ***Now try Exercise 9.***

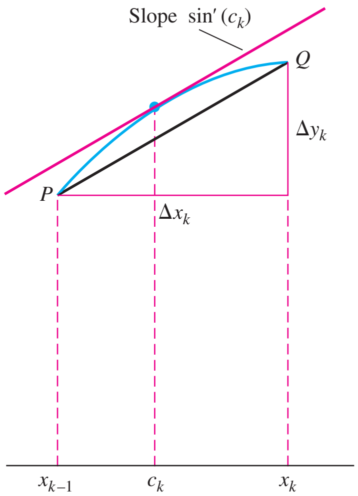


Figure 7.34 The portion of the sine curve above $[x_{k-1}, x_k]$. At some c_k in the interval, $\sin'(c_k) = \Delta y_k/\Delta x_k$, the slope of segment PQ. (Example 1)

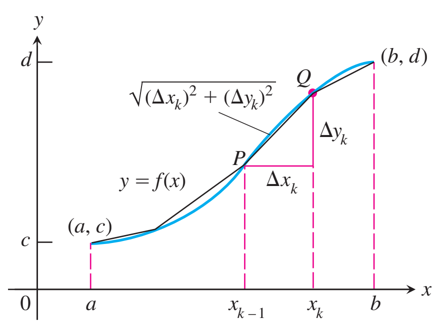


Figure 7.35 The graph of f, approximated by line segments.

Length of a Smooth Curve

We are almost ready to define the length of a curve as a definite integral, using the procedure of Example 1. We first call attention to two properties of the sine function that came into play along the way.

We obviously used *differentiability* when we invoked the Mean Value Theorem to replace $\Delta y_k/\Delta x_k$ by $\sin'(c_k)$ for some c_k in the interval $[x_{k-1}, x_k]$. Less obviously, we used the continuity of the derivative of sine in passing from $\sum \sqrt{1 + (\sin'(c_k))^2}\,\Delta x_k$ to the Riemann integral. The requirement for finding the length of a curve by this method, then, is that the function have a continuous first derivative. We call this property **smoothness.** A function with a continuous first derivative is **smooth** and its graph is a **smooth curve.**

Let us review the process, this time with a general smooth function $f(x)$. Suppose the graph of f begins at the point (a, c) and ends at (b, d), as shown in Figure 7.35. We partition the interval $a \le x \le b$ into subintervals so short that the arcs of the curve above them are nearly straight. The length of the segment approximating the arc above the subinterval $[x_{k-1}, x_k]$ is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. The sum $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ approximates the length of the curve. We apply the Mean Value Theorem to f on each subinterval to rewrite the sum as a Riemann sum,

$$\sum \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

$$= \sum \sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \qquad \text{For some point } c_k \text{ in } (x_{k-1}, x_k)$$

Passing to the limit as the norms of the subdivisions go to zero gives the length of the curve as

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We could as easily have transformed $\sum \sqrt{\Delta x_k^2 + \Delta y_k^2}$ into a Riemann sum by dividing and multiplying by Δy_k, giving a formula that involves x as a function of y (say, $x = g(y)$) on the interval $[c, d]$:

$$L \approx \sum \frac{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}}{\Delta y_k}\,\Delta y_k = \sum \sqrt{1 + \left(\frac{\Delta x_k}{\Delta y_k}\right)^2}\,\Delta y_k$$

$$= \sum \sqrt{1 + (g'(c_k))^2}\,\Delta y_k. \qquad \text{For some } c_k \text{ in } (y_{k-1}, y_k)$$

The limit of these sums, as the norms of the subdivisions go to zero, gives another reasonable way to calculate the curve's length,

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

Putting these two formulas together, we have the following definition for the length of a smooth curve.

DEFINITION Arc Length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d), $a < b$, $c < d$, then the **length (arc length) of the curve** is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b];$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d].$$

EXAMPLE 2 Applying the Definition

Find the *exact* length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \quad \text{for} \quad 0 \le x \le 1.$$

SOLUTION

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}\,x^{1/2},$$

which is continuous on $[0, 1]$. Therefore,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + \left(2\sqrt{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^1 \sqrt{1 + 8x}\,dx \\ &= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 \\ &= \frac{13}{6}. \end{aligned}$$

Now try Exercise 11.

We asked for an exact length in Example 2 to take advantage of the rare opportunity it afforded of taking the antiderivative of an arc length integrand. When you add 1 to the square of the derivative of an arbitrary smooth function and then take the square root of that sum, the result is rarely antidifferentiable by reasonable methods. We know a few more functions that give "nice" integrands, but we are saving those for the exercises.

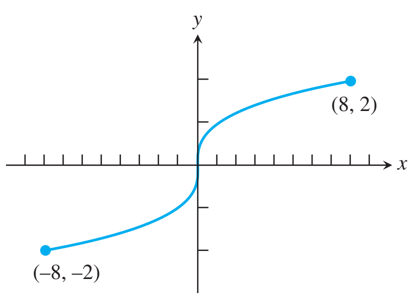


Figure 7.36 The graph of $y = x^{1/3}$ has a vertical tangent line at the origin where dy/dx does not exist. (Example 3)

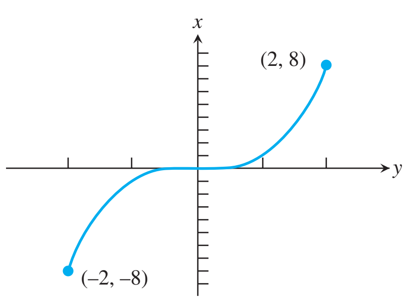


Figure 7.37 The curve in Figure 7.36 plotted with x as a function of y. The tangent at the origin is now horizontal. (Example 3)

Vertical Tangents, Corners, and Cusps

Sometimes a curve has a vertical tangent, corner, or cusp where the derivative we need to work with is undefined. We can sometimes get around such a difficulty in ways illustrated by the following examples.

EXAMPLE 3 A Vertical Tangent

Find the length of the curve $y = x^{1/3}$ between $(-8, -2)$ and $(8, 2)$.

SOLUTION

The derivative

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

is not defined at $x = 0$. Graphically, there is a vertical tangent at $x = 0$ where the derivative becomes infinite (Figure 7.36). If we change to x as a function of y, the tangent at the origin will be horizontal (Figure 7.37) and the derivative will be zero instead of undefined. Solving $y = x^{1/3}$ for x gives $x = y^3$, and we have

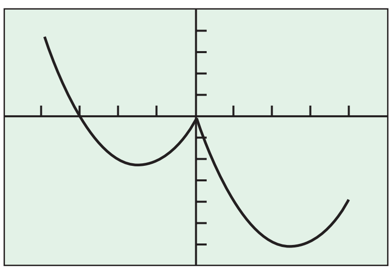
$$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{-2}^{2} \sqrt{1 + (3y^2)^2}\,dy \approx 17.26. \qquad \text{Using NINT}$$

Now try Exercise 25.

What happens if you fail to notice that dy/dx is undefined at $x = 0$ and ask your calculator to compute

$$\text{NINT}\left(\sqrt{1 + \left((1/3)\,x^{-2/3}\right)^2},\ x,\ -8,\ 8\right)?$$

This actually depends on your calculator. If, in the process of its calculations, it tries to evaluate the function at $x = 0$, then some sort of domain error will result. If it tries to find convergent Riemann sums near $x = 0$, it might get into a long, futile loop of computations that you will have to interrupt. Or it might actually produce an answer—in which case you hope it would be sufficiently bizarre for you to realize that it should not be trusted.



[–5, 5] by [–7, 5]

Figure 7.38 The graph of

$$y = x^2 - 4|x| - x,\ -4 \le x \le 4,$$

has a corner at $x = 0$ where neither dy/dx nor dx/dy exists. We find the lengths of the two smooth pieces and add them together. (Example 4)

EXAMPLE 4 Getting Around a Corner

Find the length of the curve $y = x^2 - 4|x| - x$ from $x = -4$ to $x = 4$.

SOLUTION

We should always be alert for abrupt slope changes when absolute value is involved. We graph the function to check (Figure 7.38).

There is clearly a corner at $x = 0$ where neither dy/dx nor dx/dy can exist. To find the length, we split the curve at $x = 0$ to write the function *without* absolute values:

$$x^2 - 4|x| - x = \begin{cases} x^2 + 3x & \text{if} \quad x < 0, \\ x^2 - 5x & \text{if} \quad x \ge 0. \end{cases}$$

Then,

$$L = \int_{-4}^{0} \sqrt{1 + (2x + 3)^2}\, dx + \int_{0}^{4} \sqrt{1 + (2x - 5)^2}\, dx$$

$$\approx 19.56. \qquad \text{By NINT}$$

Now try Exercise 27.

Finally, cusps are handled the same way corners are: split the curve into smooth pieces and add the lengths of those pieces.

Quick Review 7.4 *(For help, go to Sections 1.3 and 3.2.)*

In Exercises 1–5, simplify the function.

1. $\sqrt{1 + 2x + x^2}$ on $[1, 5]$ $x + 1$

2. $\sqrt{1 - x + \dfrac{x^2}{4}}$ on $[-3, -1]$ $\dfrac{2 - x}{2}$

3. $\sqrt{1 + (\tan x)^2}$ on $[0, \pi/3]$ $\sec x$

4. $\sqrt{1 + (x/4 - 1/x)^2}$ on $[4, 12]$ $\dfrac{x^2 + 4}{4x}$

5. $\sqrt{1 + \cos 2x}$ on $[0, \pi/2]$ $\sqrt{2}\cos x$

In Exercises 6–10, identify all values of x for which the function fails to be differentiable.

6. $f(x) = |x - 4|$ 4

7. $f(x) = 5x^{2/3}$ 0

8. $f(x) = \sqrt[5]{x + 3}$ -3

9. $f(x) = \sqrt{x^2 - 4x + 4}$ 2

10. $f(x) = 1 + \sqrt[3]{\sin x}$ $k\pi$, k any integer

Section 7.4 Exercises

In Exercises 1–10,

(a) set up an integral for the length of the curve;

(b) graph the curve to see what it looks like;

(c) use NINT to find the length of the curve.

1. $y = x^2, \quad -1 \le x \le 2$

2. $y = \tan x, \quad -\pi/3 \le x \le 0$

3. $x = \sin y, \quad 0 \le y \le \pi$

4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

5. $y^2 + 2y = 2x + 1,$ from $(-1, -1)$ to $(7, 3)$

6. $y = \sin x - x\cos x, \quad 0 \le x \le \pi$

7. $y = \int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$

8. $x = \int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

9. $y = \sec x, \quad -\pi/3 \le x \le \pi/3$

10. $y = (e^x + e^{-x})/2, \quad -3 \le x \le 3$

In Exercises 11–18, find the exact length of the curve analytically by antidifferentiation. You will need to simplify the integrand algebraically before finding an antiderivative.

11. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$ 12

12. $y = x^{3/2}$ from $x = 0$ to $x = 4$ $(80\sqrt{10} - 8)/27$

13. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 53/6

14. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 123/32

15. $x = (y^3/6) + 1/(2y)$ from $y = 1$ to $y = 2$
[*Hint:* $1 + (dx/dy)^2$ is a perfect square.] 17/12

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$ 53/6

17. $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$ 2

18. $y = \int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$ $7\sqrt{3}/3$

19. (a) Group Activity Find a curve through the point (1, 1) whose length integral is $y = \sqrt{x}$

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

(b) Writing to Learn How many such curves are there? Give reasons for your answer.

20. (a) Group Activity Find a curve through the point (0, 1) whose length integral is $y = 1/(1 - x)$

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

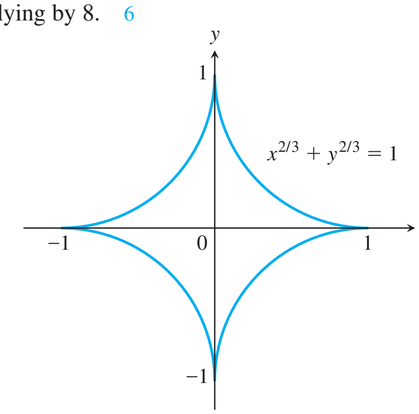
(b) Writing to Learn How many such curves are there? Give reasons for your answer.

21. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$. 1

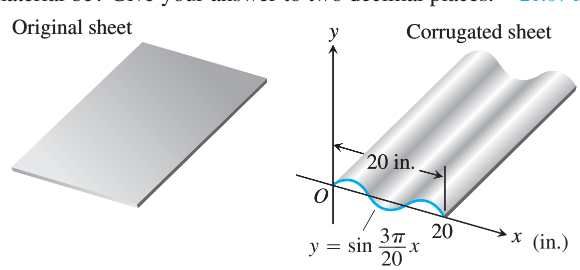
22. *The Length of an Astroid* The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of the family of curves called *astroids* (not "asteroids") because of their starlike appearance (see figure). Find the length of this particular astroid by finding the length of half the first quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8. 6



23. *Fabricating Metal Sheets* Your metal fabrication company is bidding for a contract to make sheets of corrugated steel roofing like the one shown here. The cross sections of the corrugated sheets are to conform to the curve

$$y = \sin\left(\frac{3\pi}{20}x\right), \qquad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? Give your answer to two decimal places. ≈21.07 inches

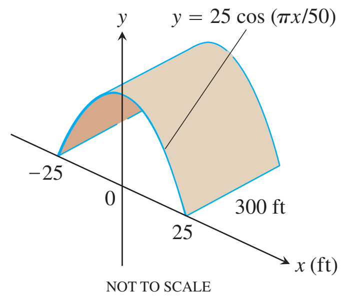


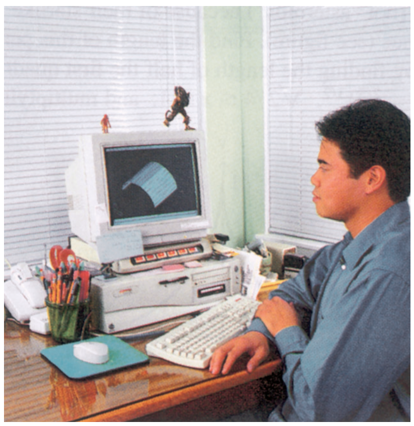
24. *Tunnel Construction* Your engineering firm is bidding for the contract to construct the tunnel shown on the next page. The tunnel is 300 ft long and 50 ft wide at the base. The cross section is shaped like one arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? \$38,422

19. (b) Only one. We know the derivative of the function and the value of the function at one value of x.

20. (b) Only one. We know the derivative of the function and the value of the function at one value of x.

30. Because the limit of the sum $\sum \Delta x_k$ as the norm of the partition goes to zero will always be the length $(b - a)$ of the interval (a, b).





In Exercises 25 and 26, find the length of the curve.

25. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$ ≈ 3.6142

26. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$ ≈ 2.1089

In Exercises 27–29, find the length of the nonsmooth curve.

27. $y = x^3 + 5|x|$ from $x = -2$ to $x = 1$ ≈ 13.132

28. $\sqrt{x} + \sqrt{y} = 1$ ≈ 1.623

29. $y = \sqrt[4]{x}$ from $x = 0$ to $x = 16$ ≈ 16.647

30. Writing to Learn Explain geometrically why it does not work to use short *horizontal* line segments to approximate the lengths of small arcs when we search for a Riemann sum that leads to the formula for arc length.

31. Writing to Learn A curve is totally contained inside the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1). Is there any limit to the possible length of the curve? Explain.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

32. True or False If a function $y = f(x)$ is continuous on an interval $[a, b]$, then the length of its curve is given by $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. Justify your answer. False. The function must be differentiable.

33. True or False If a function $y = f(x)$ is differentiable on an interval $[a, b]$, then the length of its curve is given by $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. Justify your answer. True, by definition.

31. No. Consider the curve $y = \frac{1}{3}\sin\left(\frac{1}{x}\right) + 0.5$ for $0 < x < 1$.

34. Multiple Choice Which of the following gives the best approximation of the length of the arc of $y = \cos(2x)$ from $x = 0$ to $x = \pi/4$? D

(A) 0.785 **(B)** 0.955 **(C)** 1.0 **(D)** 1.318 **(E)** 1.977

35. Multiple Choice Which of the following expressions gives the length of the graph of $x = y^3$ from $y = -2$ to $y = 2$? C

(A) $\int_{-2}^{2} (1 + y^6)\,dy$ **(B)** $\int_{-2}^{2} \sqrt{1 + y^6}\,dy$

(C) $\int_{-2}^{2} \sqrt{1 + 9y^4}\,dy$ **(D)** $\int_{-2}^{2} \sqrt{1 + x^2}\,dx$

(E) $\int_{-2}^{2} \sqrt{1 + x^4}\,dx$

36. Multiple Choice Find the length of the curve described by $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$. B

(A) $\dfrac{26}{3}$ **(B)** $\dfrac{52}{3}$ **(C)** $\dfrac{512\sqrt{2}}{15}$

(D) $\dfrac{512\sqrt{2}}{15} + 8$ **(E)** 96

37. Multiple Choice Which of the following expressions should be used to find the length of the curve $y = x^{2/3}$ from $x = -1$ to $x = 1$? A

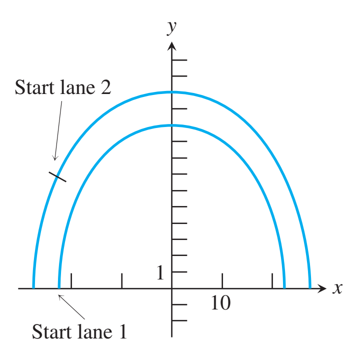
(A) $2\int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy$ **(B)** $\int_{-1}^{1} \sqrt{1 + \frac{9}{4}y}\,dy$

(C) $\int_0^1 \sqrt{1 + y^3}\,dy$ **(D)** $\int_0^1 \sqrt{1 + y^6}\,dy$

(E) $\int_0^1 \sqrt{1 + y^{9/4}}\,dy$

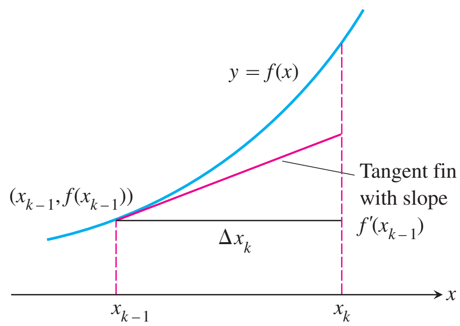
Exploration

38. *Modeling Running Tracks* Two lanes of a running track are modeled by the semiellipses as shown. The equation for lane 1 is $y = \sqrt{100 - 0.2x^2}$, and the equation for lane 2 is $y = \sqrt{150 - 0.2x^2}$. The starting point for lane 1 is at the negative x-intercept $(-\sqrt{500}, 0)$. The finish points for both lanes are the positive x-intercepts. Where should the starting point be placed on lane 2 so that the two lane lengths will be equal (running clockwise)? $\approx(-19.909, 8.410)$



Extending the Ideas

39. ***Using Tangent Fins to Find Arc Length*** Assume f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$ construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$ as shown in the figure.



(a) Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals

$$\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}.$$

(b) Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx,$$

which is the length L of the curve $y = f(x)$ from $x = a$ to $x = b$.

40. Is there a smooth curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $a\sqrt{2}$? Give reasons for your answer. Yes. Any curve of the form $y = \pm x + c$, c a constant.

39. (a) The fin is the hypotenuse of a right triangle with leg lengths Δx_k and $\left.\dfrac{df}{dx}\right|_{x=x_{k-1}} \Delta x_k = f'(x_{k-1})\,\Delta x_k$.

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}$

$\displaystyle= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(x_{k-1}))^2}$

$\displaystyle= \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

7.5 Applications from Science and Statistics

What you'll learn about

- Work Revisited
- Fluid Force and Fluid Pressure
- Normal Probabilities

. . . and why

It is important to see applications of integrals as various accumulation functions.

Our goal in this section is to hint at the diversity of ways in which the definite integral can be used. The contexts may be new to you, but we will explain what you need to know as we go along.

Work Revisited

Recall from Section 7.1 that *work* is defined as force (in the direction of motion) times displacement. A familiar example is to move against the force of gravity to lift an object. The object has to move, incidentally, before "work" is done, no matter how tired you get *trying.*

If the force $F(x)$ is not constant, then the work done in moving an object from $x = a$ to $x = b$ is the definite integral $W = \int_a^b F(x)dx$.

4.4 newtons ≈ 1 lb

(1 newton)(1 meter) = 1 N • m = 1 Joule

EXAMPLE 1 Finding the Work Done by a Force

Find the work done by the force $F(x) = \cos(\pi x)$ newtons along the x-axis from $x = 0$ meters to $x = 1/2$ meter.

SOLUTION

$$\begin{aligned} W &= \int_0^{1/2} \cos(\pi x)\, dx \\ &= \frac{1}{\pi} \sin(\pi x)\Big|_0^{1/2} \\ &= \frac{1}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) \\ &= \frac{1}{\pi} \approx 0.318 \end{aligned}$$

Now try Exercise 1.

EXAMPLE 2 Work Done Lifting

A leaky bucket weighs 22 newtons (N) empty. It is lifted from the ground at a constant rate to a point 20 m above the ground by a rope weighing 0.4 N/m. The bucket starts with 70 N (approximately 7.1 liters) of water, but it leaks at a constant rate and just finishes draining as the bucket reaches the top. Find the amount of work done

(a) lifting the bucket alone;

(b) lifting the water alone;

(c) lifting the rope alone;

(d) lifting the bucket, water, and rope together.

SOLUTION

(a) *The bucket alone.* This is easy because the bucket's weight is constant. To lift it, you must exert a force of 22 N through the entire 20-meter interval.

$$\text{Work} = (22 \text{ N}) \times (20 \text{ m}) = 440 \text{ N} \cdot \text{m} = 440 \text{ J}$$

Figure 7.39 shows the graph of force vs. distance applied. The work corresponds to the area under the force graph.

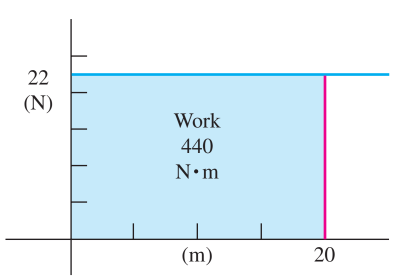


Figure 7.39 The work done by a constant 22-N force lifting a bucket 20 m is 440 N • m. (Example 2)

continued

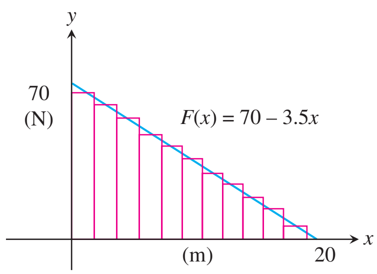


Figure 7.40 The force required to lift the water varies with distance but the work still corresponds to the area under the force graph. (Example 2)

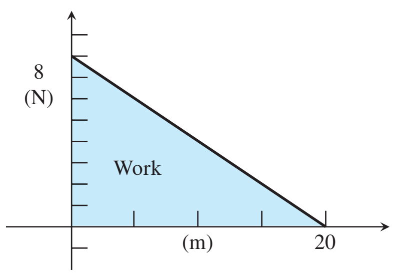


Figure 7.41 The work done lifting the rope to the top corresponds to the area of another triangle. (Example 2)

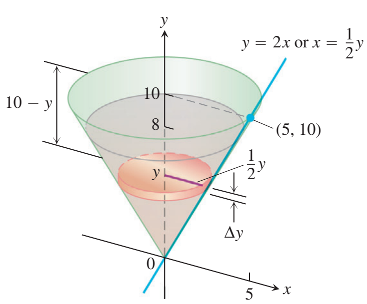


Figure 7.42 The conical tank in Example 3.

(b) *The water alone.* The force needed to lift the water is equal to the water's weight, which decreases steadily from 70 N to 0 N over the 20-m lift. When the bucket is x m off the ground, the water weighs

$$F(x) = 70\left(\frac{20-x}{20}\right) = 70\left(1 - \frac{x}{20}\right) = 70 - 3.5x \text{ N.}$$

original weight of water — proportion left at elevation x

The work done is (Figure 7.40)

$$W = \int_a^b F(x)\,dx$$

$$= \int_0^{20} (70 - 3.5x)\,dx = \Big[70x - 1.75x^2\Big]_0^{20} = 1400 - 700 = 700 \text{ J.}$$

(c) *The rope alone.* The force needed to lift the rope is also variable, starting at $(0.4)(20) = 8$ N when the bucket is on the ground and ending at 0 N when the bucket and rope are all at the top. As with the leaky bucket, the rate of decrease is constant. At elevation x meters, the $(20 - x)$ meters of rope still there to lift weigh $F(x) = (0.4)(20 - x)$ N. Figure 7.41 shows the graph of F. The work done lifting the rope is

$$\int_0^{20} F(x)\,dx = \int_0^{20} (0.4)(20 - x)\,dx$$

$$= \Big[8x - 0.2x^2\Big]_0^{20} = 160 - 80 = 80 \text{ N}\cdot\text{m} = 80 \text{ J.}$$

(d) *The bucket, water, and rope together.* The total work is

$$440 + 700 + 80 = 1220 \text{ J.}$$

Now try Exercise 5.

EXAMPLE 3 Work Done Pumping

The conical tank in Figure 7.42 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

SOLUTION

We imagine the oil partitioned into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. (The 8 represents the top of the oil, not the top of the tank.)

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y \ \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4}y^2\,\Delta y \ \text{ lb.} \qquad \text{Weight} = \left(\begin{matrix}\text{weight per}\\ \text{unit volume}\end{matrix}\right) \times \text{volume}$$

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2\,\Delta y \ \text{ ft}\cdot\text{lb.}$$

The work done lifting all the slabs from $y = 0$ to $y = 8$ to the rim is approximately

$$W \approx \sum \frac{57\pi}{4}(10 - y)\,y^2\,\Delta y \ \text{ ft}\cdot\text{lb.}$$

continued

This is a Riemann sum for the function $(57\pi/4)(10 - y)y^2$ on the interval from $y = 0$ to $y = 8$. The work of pumping the oil to the rim is the limit of these sums as the norms of the partitions go to zero.

$$W = \int_0^8 \frac{57\pi}{4}(10 - y)y^2\,dy = \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\,dy$$

$$= \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 30{,}561 \text{ ft} \cdot \text{lb}$$

Now try Exercise 17.

Fluid Force and Fluid Pressure

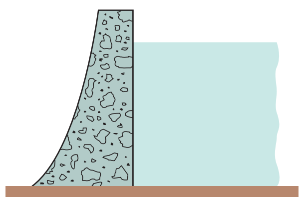


Figure 7.43 To withstand the increasing pressure, dams are built thicker toward the bottom.

We make dams thicker at the bottom than at the top (Figure 7.43) because the pressure against them increases with depth. It is a remarkable fact that the pressure at any point on a dam depends only on how far below the surface the point lies and not on how much water the dam is holding back. In any liquid, the **fluid pressure** p (force per unit area) at depth h is

$$p = wh, \qquad \text{Dimensions check: } \frac{\text{lb}}{\text{ft}^2} = \frac{\text{lb}}{\text{ft}^3} \times \text{ft, for example}$$

where w is the *weight-density* (weight per unit volume) of the liquid.

Typical Weight-densities (lb/ft^3)

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Seawater	64
Water	62.4

EXAMPLE 4 The Great Molasses Flood of 1919

At 1:00 P.M. on January 15, 1919 (an unseasonably warm day), a 90-ft-high, 90-foot-diameter cylindrical metal tank in which the Puritan Distilling Company stored molasses at the corner of Foster and Commercial streets in Boston's North End exploded. Molasses flooded the streets 30 feet deep, trapping pedestrians and horses, knocking down buildings, and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs via trolley cars and people's shoes. It took weeks to clean up.

(a) Given that the tank was full of molasses weighing 100 lb/ft^3, what was the total force exerted by the molasses on the bottom of the tank at the time it ruptured?

(b) What was the total force against the bottom foot-wide band of the tank wall (Figure 7.44)?

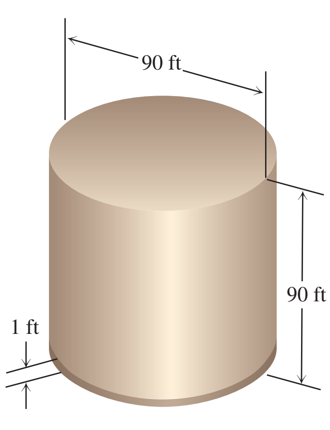


Figure 7.44 The molasses tank of Example 4.

continued

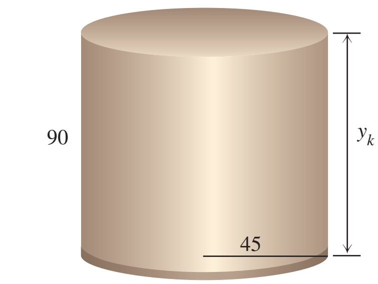


Figure 7.45 The 1-ft band at the bottom of the tank wall can be partitioned into thin strips on which the pressure is approximately constant. (Example 4)

SOLUTION

(a) At the bottom of the tank, the molasses exerted a constant pressure of

$$p = wh = \left(100\,\frac{\text{lb}}{\text{ft}^3}\right)(90\text{ ft}) = 9000\,\frac{\text{lb}}{\text{ft}^2}.$$

Since the area of the base was $\pi(45)^2$, the total force on the base was

$$\left(9000\,\frac{\text{lb}}{\text{ft}^2}\right)(2025\,\pi\text{ ft}^2) \approx 57{,}225{,}526\text{ lb}.$$

(b) We partition the band from depth 89 ft to depth 90 ft into narrower bands of width Δy and choose a depth y_k in each one. The pressure at this depth y_k is $p = wh = 100\,y_k$ lb/ft^2 (Figure 7.45). The force against each narrow band is approximately

$$\text{pressure} \times \text{area} = (100y_k)(90\pi\,\Delta y) = 9000\pi\,y_k\,\Delta y\text{ lb}.$$

Adding the forces against all the bands in the partition and passing to the limit as the norms go to zero, we arrive at

$$F = \int_{89}^{90} 9000\pi y\,dy = 9000\pi\int_{89}^{90} y\,dy \approx 2{,}530{,}553\text{ lb}$$

for the force against the bottom foot of tank wall. *Now try Exercise 25.*

Normal Probabilities

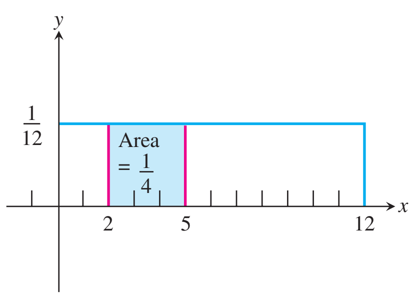


Figure 7.46 The probability that the clock has stopped between 2:00 and 5:00 can be represented as an area of 1/4. The rectangle over the entire interval has area 1.

Suppose you find an old clock in the attic. What is the probability that it has stopped somewhere between 2:00 and 5:00?

If you imagine time being measured continuously over a 12-hour interval, it is easy to conclude that the answer is 1/4 (since the interval from 2:00 to 5:00 contains one-fourth of the time), and that is correct. Mathematically, however, the situation is not quite that clear because both the 12-hour interval and the 3-hour interval contain an *infinite* number of times. In what sense does the ratio of one infinity to another infinity equal 1/4?

The easiest way to resolve that question is to look at area. We represent the total probability of the 12-hour interval as a rectangle of area 1 sitting above the interval (Figure 7.46).

Not only does it make perfect sense to say that the rectangle over the time interval $[2, 5]$ has an area that is one-fourth the area of the total rectangle, the area actually *equals* 1/4, since the total rectangle has area 1. That is why mathematicians represent probabilities as areas, and that is where definite integrals enter the picture.

Improper Integrals

More information about improper integrals like $\int_{-\infty}^{\infty} f(x)\,dx$ can be found in Section 8.3. (You will not need that information here.)

DEFINITION Probability Density Function (pdf)

A **probability density function** is a function $f(x)$ with domain all reals such that

$$f(x) \geq 0 \text{ for all } x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Then the probability associated with an interval $[a, b]$ is

$$\int_a^b f(x)\,dx.$$

Probabilities of events, such as the clock stopping between 2:00 and 5:00, are integrals of an appropriate pdf.

EXAMPLE 5 Probability of the Clock Stopping

Find the probability that the clock stopped between 2:00 and 5:00.

SOLUTION

The pdf of the clock is

$$f(t) = \begin{cases} 1/12, & 0 \le t \le 12 \\ 0, & \text{otherwise.} \end{cases}$$

The probability that the clock stopped at some time t with $2 \le t \le 5$ is

$$\int_2^5 f(t)\,dt = \frac{1}{4}.$$

Now try Exercise 27.

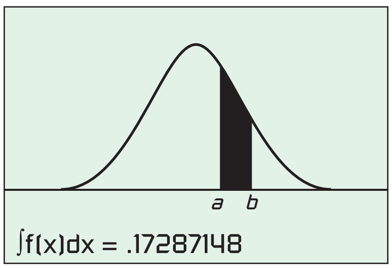


Figure 7.47 A normal probability density function. The probability associated with the interval $[a, b]$ is the area under the curve, as shown.

By far the most useful kind of pdf is the *normal* kind. ("Normal" here is a technical term, referring to a curve with the shape in Figure 7.47.) The **normal curve,** often called the "bell curve," is one of the most significant curves in applied mathematics because it enables us to describe entire populations based on the statistical measurements taken from a reasonably-sized sample. The measurements needed are the *mean* (μ) and the *standard deviation* (σ), which your calculators will approximate for you from the data. The symbols on the calculator will probably be $\overline{x}$ and s (see your *Owner's Manual*), but go ahead and use them as μ and σ, respectively. Once you have the numbers, you can find the curve by using the following remarkable formula discovered by Karl Friedrich Gauss.

DEFINITION Normal Probability Density Function (pdf)

The **normal probability density function (Gaussian curve)** for a population with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$

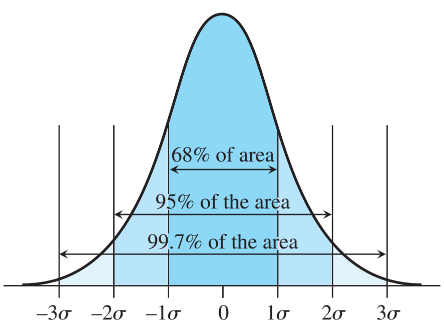


Figure 7.48 The 68-95-99.7 rule for normal distributions.

The mean μ represents the average value of the variable x. The standard deviation σ measures the "scatter" around the mean. For a normal curve, the mean and standard deviation tell you where most of the probability lies. The rule of thumb, illustrated in Figure 7.48, is this:

The 68-95-99.7 Rule for Normal Distributions

Given a normal curve,

- 68% of the area will lie within σ of the mean μ,
- 95% of the area will lie within 2σ of the mean μ,
- 99.7% of the area will lie within 3σ of the mean μ.

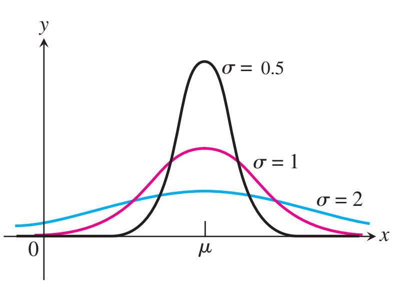


Figure 7.49 Normal pdf curves with mean $\mu = 2$ and $\sigma = 0.5$, 1, and 2.

Even with the 68-95-99.7 rule, the area under the curve can spread quite a bit, depending on the size of σ. Figure 7.49 shows three normal pdfs with mean $\mu = 2$ and standard deviations equal to 0.5, 1, and 2.

EXAMPLE 6 A Telephone Help Line

Suppose a telephone help line takes a mean of 2 minutes to answer calls. If the standard deviation is $\sigma = 0.5$, then 68% of the calls are answered in the range of 1.5 to 2.5 minutes and 99.7% of the calls are answered in the range of 0.5 to 3.5 minutes.

Now try Exercise 29.

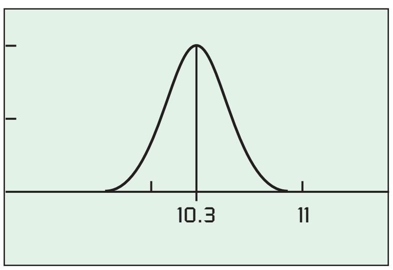
EXAMPLE 7 Weights of Spinach Boxes

Suppose that frozen spinach boxes marked as "10 ounces" of spinach have a mean weight of 10.3 ounces and a standard deviation of 0.2 ounce.

(a) What percentage of *all* such spinach boxes can be expected to weigh between 10 and 11 ounces?

(b) What percentage would we expect to weigh less than 10 ounces?

(c) What is the probability that a box weighs *exactly* 10 ounces?



[9, 11.5] by [–1, 2.5]

Figure 7.50 The normal pdf for the spinach weights in Example 7. The mean is at the center.

SOLUTION

Assuming that some person or machine is *trying* to pack 10 ounces of spinach into these boxes, we expect that most of the weights will be around 10, with probabilities tailing off for boxes being heavier or lighter. We expect, in other words, that a normal pdf will model these probabilities. First, we define $f(x)$ using the formula:

$$f(x) = \frac{1}{0.2\sqrt{2\pi}} e^{-(x-10.3)^2/(0.08)}.$$

The graph (Figure 7.50) has the look we are expecting.

(a) For an arbitrary box of this spinach, the probability that it weighs between 10 and 11 ounces is the area under the curve from 10 to 11, which is

$$\text{NINT}\,(f(x), x, 10, 11) \approx 0.933.$$

So without doing any more measuring, we can predict that about 93.3% of all such spinach boxes will weigh between 10 and 11 ounces.

(b) For the probability that a box weighs less than 10 ounces, we use the entire area under the curve to the left of $x = 10$. The curve actually approaches the x-axis as an asymptote, but you can see from the graph (Figure 7.50) that $f(x)$ approaches zero quite quickly. Indeed, $f(9)$ is only slightly larger than a billionth. So getting the area from 9 to 10 should do it:

$$\text{NINT}\,(f(x), x, 9, 10) \approx 0.067.$$

We would expect only about 6.7% of the boxes to weigh less than 10 ounces.

(c) This would be the integral from 10 to 10, which is zero. This zero probability might seem strange at first, but remember that we are assuming a continuous, unbroken interval of possible spinach weights, and 10 is but one of an infinite number of them.

Now try Exercise 31.

Quick Review 7.5 *(For help, go to Section 5.2.)*

In Exercises 1–5, find the definite integral by **(a)** antiderivatives and **(b)** using NINT.

1. $\int_0^1 e^{-x}\,dx$ **a.** $1 - (1/e)$ **b.** ≈ 0.632

2. $\int_0^1 e^x\,dx$ **a.** $e - 1$ **b.** ≈ 1.718

3. $\int_{\pi/4}^{\pi/2} \sin x\,dx$ **a.** $\sqrt{2}/2$ **b.** ≈ 0.707

4. $\int_0^3 (x^2 + 2)\,dx$ 15

5. $\int_1^2 \frac{x^2}{x^3 + 1}\,dx$ **a.** $(1/3)\ln(9/2)$ **b.** ≈ 0.501

In Exercises 6–10 find, but do not evaluate, the definite integral that is the limit as the norms of the partitions go to zero of the Riemann sums on the closed interval [0, 7].

6. $\sum 2\pi(x_k + 2)(\sin x_k)\Delta x$ $\int_0^7 2\pi(x + 2)\sin x\, dx$

7. $\sum (1 - x_k^2)(2\pi x_k)\Delta x$ $\int_0^7 (1 - x^2)(2\pi x)\, dx$

8. $\sum \pi(\cos x_k)^2 \Delta x$ $\int_0^7 \pi \cos^2 x\, dx$

9. $\sum \pi\left(\frac{y_k}{2}\right)^2 (10 - y_k)\Delta y$ $\int_0^7 \pi(y/2)^2 (10 - y)\, dy$

10. $\sum \frac{\sqrt{3}}{4}(\sin^2 x_k)\Delta x$ $\int_0^7 (\sqrt{3}/4)\sin^2 x\, dx$

Section 7.5 Exercises

In Exercises 1–4, find the work done by the force of $F(x)$ newtons along the x-axis from $x = a$ meters to $x = b$ meters.

1. $F(x) = xe^{-x/3}$, $a = 0$, $b = 5$ ≈4.4670 J

2. $F(x) = x\sin(\pi x/4)$, $a = 0$, $b = 3$ ≈3.8473 J

3. $F(x) = x\sqrt{9 - x^2}$, $a = 0$, $b = 3$ 9 J

4. $F(x) = e^{\sin x}\cos x + 2$, $a = 0$, $b = 10$ ≈19.5804 J

5. ***Leaky Bucket*** The workers in Example 2 changed to a larger bucket that held 50 L (490 N) of water, but the new bucket had an even larger leak so that it too was empty by the time it reached the top. Assuming the water leaked out at a steady rate, how much work was done lifting the water to a point 20 meters above the ground? (Do not include the rope and bucket.) 4900 J

6. ***Leaky Bucket*** The bucket in Exercise 5 is hauled up more quickly so that there is still 10 L (98 N) of water left when the bucket reaches the top. How much work is done lifting the water this time? (Do not include the rope and bucket.) 5880 J

7. ***Leaky Sand Bag*** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 18 ft. How much work was done lifting the sand this far? (Neglect the weights of the bag and lifting equipment.) 1944 ft-lb

8. ***Stretching a Spring*** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.

(a) Find the force constant. 200 lb/in.

(b) How much work is done in stretching the spring from 10 in. to 12 in.? 400 in.-lb

(c) How far beyond its natural length will a 1600-lb force stretch the spring? 8 in.

9. ***Subway Car Springs*** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

(a) What is the assembly's force constant? 7238 lb/in.

(b) How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest inch-pound. ≈905 in.-lb and ≈2714 in.-lb

(*Source:* Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

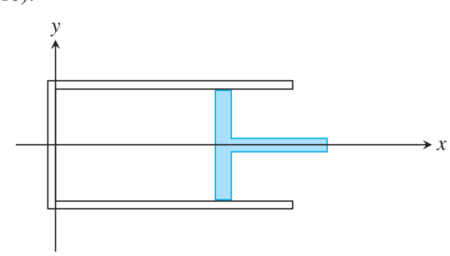
10. ***Bathroom Scale*** A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming the scale behaves like a spring that obeys Hooke's Law,

(a) how much does someone who compresses the scale 1/8 in. weigh? 300 lb

(b) how much work is done in compressing the scale 1/8 in.? 18.75 in.-lb

11. ***Hauling a Rope*** A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m? 780 J

12. ***Compressing Gas*** Suppose that gas in a circular cylinder of cross section area A is being compressed by a piston (see figure).



(a) If p is the pressure of the gas in pounds per square inch and V is the volume in cubic inches, show that the work done in compressing the gas from state (p_1, V_1) to state (p_2, V_2) is given by the equation

$$\text{Work} = \int_{(p_1, V_1)}^{(p_2, V_2)} p\, dV \quad \text{in.} \cdot \text{lb},$$

where the force against the piston is pA.

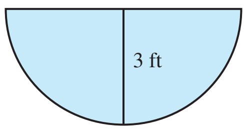
(b) Find the work done in compressing the gas from $V_1 = 243$ in^3 to $V_2 = 32$ in^3 if $p_1 = 50$ lb/in^3 and p and V obey the gas law $pV^{1.4}$ = constant (for adiabatic processes). –37,968.75 in.-lb

Group Activity In Exercises 13–16, the vertical end of a tank containing water (blue shading) weighing 62.4 lb/ft^3 has the given shape.

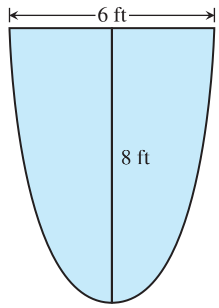
(a) **Writing to Learn** Explain how to approximate the force against the end of the tank by a Riemann sum.

(b) Find the force as an integral and evaluate it.

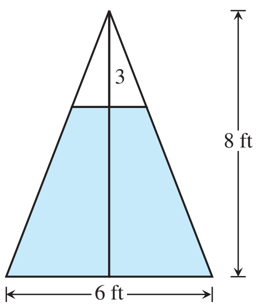
13. semicircle **(b)** 1123.2 lb



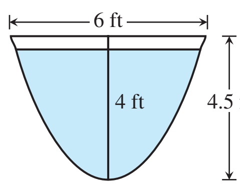
14. semiellipse **(b)** 7987.2 lb



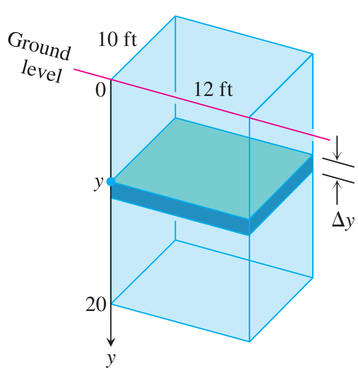
15. triangle **(b)** 3705 lb



16. parabola **(b)** ≈1506.1 lb



17. ***Pumping Water*** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.



(a) How much work does it take to empty the tank by pumping the water back to ground level once the tank is full? 1,497,600 ft-lb

(b) If the water is pumped to ground level with a (5/11)-horsepower motor (work output 250 ft • lb/sec), how long will it take to empty the full tank (to the nearest minute)? ≈100 min

(c) Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.

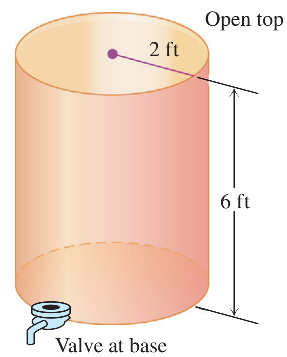
17. (d) 1,494,240 ft-lb, ≈100 min; 1,500,000 ft-lb, 100 min

(d) ***The Weight of Water*** Because of differences in the strength of Earth's gravitational field, the weight of a cubic foot of water at sea level can vary from as little as 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot of water that weighs 62.4 lb in Melbourne or New York City will weigh 62.5 lb in Juneau or Stockholm. What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.5 lb/ft^3?

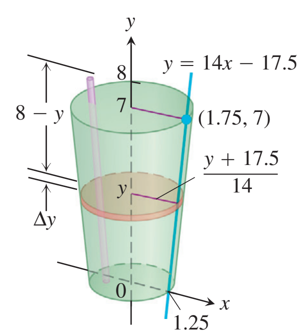
18. ***Emptying a Tank*** A vertical right cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank? ≈7,238,229 ft-lb

19. **Writing to Learn** The cylindrical tank shown here is to be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about this. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will require less work? Give reasons for your answer.

Through valve: ≈84,687.3 ft-lb
Over the rim: ≈98,801.8 ft-lb
Through a hose attached to a valve in the bottom is faster, because it takes more time to do more work.



20. ***Drinking a Milkshake*** The truncated conical container shown here is full of strawberry milkshake that weighs (4/9) oz/in^3. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to drink the milkshake through the straw (neglecting friction)? Answer in inch-ounces. ≈91.3244 in.-oz

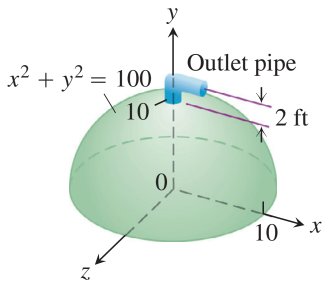


Dimensions in inches

21. ***Revisiting Example 3*** How much work will it take to pump the oil in Example 3 to a level 3 ft above the cone's rim? ≈53,482.5 ft-lb

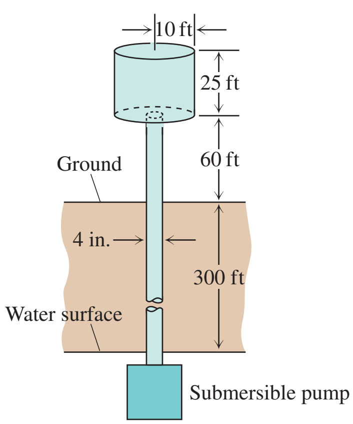
22. ***Pumping Milk*** Suppose the conical tank in Example 3 contains milk weighing 64.5 lb/ft^3 instead of olive oil. How much work will it take to pump the contents to the rim? $\approx$34,582.65 ft-lb

23. **Writing to Learn** You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3.



A firm you contacted says it can empty the tank for 1/2 cent per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the tank. If you have budgeted $5000 for the job, can you afford to hire the firm? $\approx$967,611 ft-lb, yes

24. ***Water Tower*** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300-ft well through a vertical 4-in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft above ground. The pump is a 3-hp pump, rated at 1650 ft • lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume water weighs 62.4 lb/ft^3. $\approx$31 hr



NOT TO SCALE

25. ***Fish Tank*** A rectangular freshwater fish tank with base 2×4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.

(a) Find the fluid force against each end of the tank. $\approx$209.73 lb

(b) Suppose the tank is sealed and stood on end (without spilling) so that one of the square ends is the base. What does that do to the fluid forces on the rectangular sides? $\approx$838.93 lb; the fluid force doubles

26. ***Milk Carton*** A rectangular milk carton measures 3.75 in. by 3.75 in. at the base and is 7.75 in. tall. Find the force of the milk (weighing 64.5 lb/ft^3) on one side when the carton is full. $\approx$4.2 lb

27. Find the probability that a clock stopped between 1:00 and 5:00. 1/3

28. Find the probability that a clock stopped between 3:00 and 6:00. 1/4

29. Suppose a telephone help line takes a mean of 2 minutes to answer calls. If the standard deviation is $\sigma = 2$, what percentage of the calls are answered in the range of 0 to 4 minutes? 68%

30. ***Test Scores*** The mean score on a national aptitude test is 498 with a standard deviation of 100 points.

(a) What percentage of the population has scores between 400 and 500? $\approx$0.34 (34%)

(b) If we sample 300 test-takers at random, about how many should have scores above 700? 6.5

31. ***Heights of Females*** The mean height of an adult female in New York City is estimated to be 63.4 inches with a standard deviation of 3.2 inches. What proportion of the adult females in New York City are

(a) less than 63.4 inches tall? 0.5 (50%)

(b) between 63 and 65 inches tall? $\approx$0.24 (24%)

(c) taller than 6 feet? $\approx$0.0036 (0.36%)

(d) exactly 5 feet tall? 0 if we assume a continuous distribution; $\approx$0.071; 7.1% between 59.5 in. and 60.5 in.

32. **Writing to Learn** Exercises 30 and 31 are subtly different, in that the heights in Exercise 31 are measured *continuously* and the scores in Exercise 30 are measured *discretely*. The discrete probabilities determine rectangles above the individual test scores, so that there actually is a nonzero probability of scoring, say, 560. The rectangles would look like the figure below, and would have total area 1.

Explain why integration gives a good estimate for the probability, even in the discrete case. Integration is a good approximation to the area.

33. **Writing to Learn** Suppose that $f(t)$ is the probability density function for the lifetime of a certain type of lightbulb where t is in hours. What is the meaning of the integral

$$\int_{100}^{800} f(t)\,dt?$$

The proportion of lightbulbs that last between 100 and 800 hours.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

34. **True or False** A force is applied to compress a spring several inches. Assume the spring obeys Hooke's Law. Twice as much work is required to compress the spring the second inch than is required to compress the spring the first inch. Justify your answer. False. Three times as much work is required.

35. True. The force against each vertical side is 842.4 lb

35. True or False An aquarium contains water weighing 62.4 lb/ft^3. The aquarium is in the shape of a cube where the length of each edge is 3 ft. Each side of the aquarium is engineered to withstand 1000 pounds of force. This should be sufficient to withstand the force from water pressure. Justify your answer.

36. Multiple Choice A force of $F(x) = 350x$ newtons moves a particle along a line from $x = 0$ m to $x = 5$ m. Which of the following gives the best approximation of the work done by the force? E

(A) 1750 J **(B)** 2187.5 J **(C)** 2916.67 J

(D) 3281.25 J **(E)** 4375 J

37. Multiple Choice A leaky bag of sand weighs 50 n. It is lifted from the ground at a constant rate, to a height of 20 m above the ground. The sand leaks at a constant rate and just finishes draining as the bag reaches the top. Which of the following gives the work done to lift the sand to the top? (Neglect the bag.) D

(A) 50 J **(B)** 100 J **(C)** 250 J **(D)** 500 J **(E)** 1000 J

38. Multiple Choice A spring has a natural length of 0.10 m. A 200-n force stretches the spring to a length of 0.15 m. Which of the following gives the work done in stretching the spring from 0.10 m to 0.15 m? B

(A) 0.05 J **(B)** 5 J **(C)** 10 J **(D)** 200 J **(E)** 4000 J

39. Multiple Choice A vertical right cylindrical tank measures 12 ft high and 16 ft in diameter. It is full of water weighing 62.4 lb/ft^3. How much work does it take to pump the water to the level of the top of the tank? Round your answer to the nearest ft-lb. E

(A) 149,490 ft-lb

(B) 285,696 ft-lb

(C) 360,240 ft-lb

(D) 448,776 ft-lb

(E) 903,331 ft-lb

Extending the Ideas

40. *Putting a Satellite into Orbit* The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6726 \times 10^{-11}\ \mathrm{N \cdot m^2 kg^{-2}}$ is the *universal gravitational constant,* and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \text{ joules.}$$

The lower limit of integration is Earth's radius in meters at the launch site. Evaluate the integral. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.) 5.1446×10^{10} J

41. *Forcing Electrons Together* Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newton.}$$

(a) Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin? 1.15×10^{-28} J

(b) Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)? $\approx 7.6667 \times 10^{-29}$ J

42. *Kinetic Energy* If a variable force of magnitude $F(x)$ moves a body of mass m along the x-axis from x_1 to x_2, the body's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion, $F = m(dv/dt)$, and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$ See page 429.

to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2, \qquad (1)$$

where v_1 and v_2 are the body's velocities at x_1 and x_2. In physics the expression $(1/2)mv^2$ is the *kinetic energy* of the body moving with velocity v. Therefore, *the work done by the force equals the change in the body's kinetic energy,* and we can find the work by calculating this change.

Weight vs. Mass

Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{weight} = \text{mass} \times \text{acceleration}.$$

Thus,

$$\text{newtons} = \text{kilograms} \times \text{m/sec}^2,$$

$$\text{pounds} = \text{slugs} \times \text{ft/sec}^2.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity.

In Exercises 43–49, use Equation 1 from Exercise 42.

43. *Tennis* A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? 50 ft-lb

44. *Baseball* How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz = 0.3125 lb. ≈85.1 ft-lb

45. *Golf* A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done getting the ball into the air? 122.5 ft-lb

46. *Tennis* During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed? ≈64.6 ft-lb

47. ***Football*** A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to that speed? ≈ 109.7 ft-lb

48. ***Softball*** How much work has to be performed on a 6.5-oz softball to pitch it at 132 ft/sec (90 mph)? ≈ 110.6 ft-lb

42. $F = m\frac{dv}{dt} = mv\frac{dv}{dx}$, so $W = \int_{x_1}^{x_2} F(x)\,dx$

$$= \int_{x_1}^{x_2} mv\frac{dv}{dx}\,dx = \int_{v_1}^{v_2} mv\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

49. ***A Ball Bearing*** A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 3 in. and released. About how high does the ball bearing go? (*Hint:* The kinetic (compression) energy, mgh, of a spring is $\frac{1}{2}ks^2$, where s is the distance the spring is compressed, m is the mass, g is the acceleration of gravity, and h is the height.) 4.5 ft

Quick Quiz for AP* Preparation: Sections 7.4 and 7.5

You should solve the following problems without using a graphing calculator.

1. **Multiple Choice** The length of a curve from $x = 0$ to $x = 1$ is given by $\int_0^1 \sqrt{1 + 16x^6}\,dx$. If the curve contains the point (1, 4), which of the following could be an equation for this curve? A

(A) $y = x^4 + 3$

(B) $y = x^4 + 1$

(C) $y = 1 + 16x^6$

(D) $y = \sqrt{1 + 16x^6}$

(E) $y = x + \frac{x^7}{7}$

2. **Multiple Choice** Which of the following gives the length of the path described by the parametric equations $x = \frac{1}{4}t^4$ and $y = t^3$, where $0 \le t \le 2$? D

(A) $\int_0^2 t^6 + 9t^4\,dt$

(B) $\int_0^2 \sqrt{t^6 + 1}\,dt$

(C) $\int_0^2 \sqrt{1 + 9t^4}\,dt$

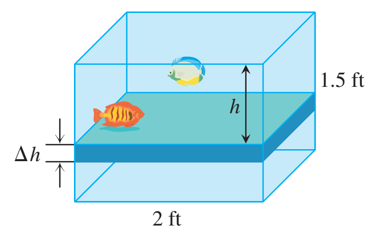
(D) $\int_0^2 \sqrt{t^6 + 9t^4}\,dt$

(E) $\int_0^2 \sqrt{t^3 + 3t^2}\,dt$

3. **Multiple Choice** The base of a solid is a circle of radius 2 inches. Each cross section perpendicular to a certain diameter is a square with one side lying in the circle. The volume of the solid in cubic inches is C

(A) 16 **(B)** 16π **(C)** $\frac{128}{3}$ **(D)** $\frac{128\pi}{3}$ **(E)** 32π

4. **Free Response** The front of a fish tank is rectangular in shape and measures 2 ft wide by 1.5 ft tall. The water in the tank exerts pressure on the front of the tank. The pressure at any point on the front of the tank depends only on how far below the surface the point lies and is given by the equation $p = 62.4h$, where h is depth below the surface measured in feet and p is pressure measured in pounds/ft^2.



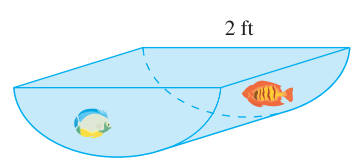
The front of the tank can be partitioned into narrow horizontal bands of height Δh. The force exerted by the water on a band at depth h_i is approximately

$$\text{pressure} \cdot \text{area} = 62.4h_i \cdot 2\Delta h.$$

(a) Write the Riemann sum that approximates the force exerted on the entire front of the tank.

(b) Use the Riemann sum from part (a) to write and evaluate a definite integral that gives the force exerted on the front of the tank. Include correct units.

(c) Find the total force exerted on the front of the tank if the front (and back) are semicircles with diameter 2 ft. Include correct units.



(a) $\sum_{i=1}^{n} 62.4h_i \cdot 2\,\Delta h$

(b) $\int_0^{1.5} 62.4h \cdot 2\,dh = 140.4$ lbs

(c) $\int_0^{1.5} 62.4h \cdot 2\sqrt{1-h^2}\,dh = 41.6$ lbs

Calculus at Work

I am working toward becoming an archeaoastronomer and ethnoastronomer of Africa. I have a Bachelor's degree in Physics, a Master's degree in Astronomy, and a Ph.D. in Astronomy and Astrophysics. From 1988 to 1990 I was a member of the Peace Corps, and I taught mathematics to high school students in the Fiji Islands. Calculus is a required course in high schools there.

For my Ph.D. dissertation, I investigated the possibility of the birthrate of stars being related to the composition of star formation clouds. I collected data on the absorption of electromagnetic emissions emanating from these regions. The intensity of emissions graphed versus wavelength produces a flat curve with downward spikes at the characteristic wavelengths of the elements present. An estimate of the area between a spike and the flat curve results in a concentration in molecules/cm^3 of an element. This area is the difference in the integrals of the flat and spike curves. In particular, I was looking for a large concentration of water-ice, which increases the probability of planets forming in a region.

Currently, I am applying for two research grants. One will allow me to use the NASA infrared telescope on Mauna Kea to search for C_3S_2 in comets. The other will help me study the history of astronomy in Tunisia.

Jarita Holbrook
Los Angeles, CA

Chapter 7 Key Terms

arc length (p. 413)
area between curves (p. 390)
Cavalieri's theorems (p. 404)
center of mass (p. 389)
constant-force formula (p. 384)
cylindrical shells (p. 402)
displacement (p. 380)
fluid force (p. 421)
fluid pressure (p. 421)
foot-pound (p. 384)
force constant (p. 385)
Gaussian curve (p. 423)
Hooke's Law (p. 385)
inflation rate (p. 388)
joule (p. 384)
length of a curve (p. 413)
mean (p. 423)
moment (p. 389)
net change (p. 379)
newton (p. 384)
normal curve (p. 423)
normal pdf (p. 423)
probability density function (pdf) (p. 422)
68-95-99.7 rule (p. 423)
smooth curve (p. 413)
smooth function (p. 413)
solid of revolution (p. 400)
standard deviation (p. 423)
surface area (p. 405)
total distance traveled (p. 381)
universal gravitational constant (p. 428)
volume by cylindrical shells (p. 402)
volume by slicing (p. 400)
volume of a solid (p. 399)
weight-density (p. 421)
work (p. 384)

Chapter 7 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–5, the application involves the accumulation of small changes over an interval to give the net change over that entire interval. Set up an integral to model the accumulation and evaluate it to answer the question.

1. A toy car slides down a ramp and coasts to a stop after 5 sec. Its velocity from $t = 0$ to $t = 5$ is modeled by $v(t) = t^2 - 0.2t^3$ ft/sec. How far does it travel? ≈ 10.417 ft

2. The fuel consumption of a diesel motor between weekly maintenance periods is modeled by the function $c(t) = 4 + 0.001t^4$ gal/day, $0 \le t \le 7$. How many gallons does it consume in a week? ≈ 31.361 gal

3. The number of billboards per mile along a 100-mile stretch of an interstate highway approaching a certain city is modeled by the function $B(x) = 21 - e^{0.03x}$, where x is the distance from the city in miles. About how many billboards are along that stretch of highway? ≈ 1464

4. A 2-meter rod has a variable density modeled by the function $\rho(x) = 11 - 4x$ g/m, where x is the distance in meters from the base of the rod. What is the total mass of the rod? 14 g

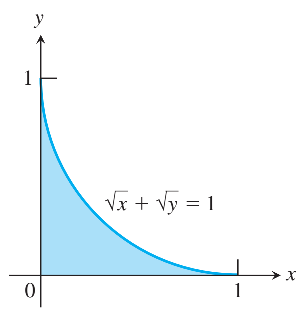
5. The electrical power consumption (measured in kilowatts) at a factory t hours after midnight during a typical day is modeled by $E(t) = 300(2 - \cos(\pi t/12))$. How many kilowatt-hours of electrical energy does the company consume in a typical day? 14,400

In Exercises 6–19, find the area of the region enclosed by the lines and curves.

6. $y = x, \quad y = 1/x^2, \quad x = 2$ 1

7. $y = x + 1, \quad y = 3 - x^2$ $\frac{9}{2}$

8. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$ 1/6



9. $x = 2y^2, \quad x = 0, \quad y = 3$ 18

10. $4x = y^2 - 4, \quad 4x = y + 16$ 30.375

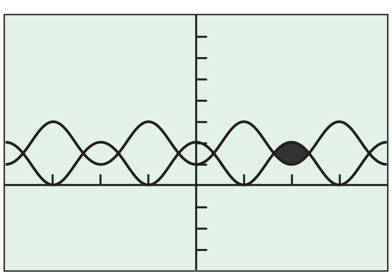
11. $y = \sin x, \quad y = x, \quad x = \pi/4$ ≈0.0155

12. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$ 4

13. $y = \cos x, \quad y = 4 - x^2$ ≈8.9023

14. $y = \sec^2 x, \quad y = 3 - |x|$ ≈2.1043

15. ***The Necklace*** one of the smaller bead-shaped regions enclosed by the graphs of $y = 1 + \cos x$ and $y = 2 - \cos x$ $2\sqrt{3} - 2\pi/3 \approx 1.370$



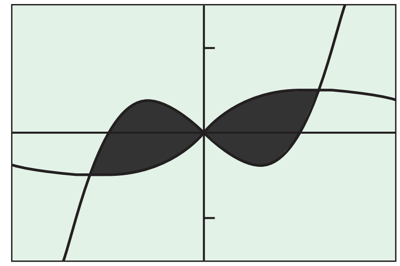
$[-4\pi, 4\pi]$ by $[-4, 8]$

16. one of the larger bead-shaped regions enclosed by the curves in Exercise 15 $2\sqrt{3} + 4\pi/3 \approx 7.653$

17. ***The Bow Tie*** the region enclosed by the graphs of

$$y = x^3 - x \quad \text{and} \quad y = \frac{x}{x^2 + 1}$$

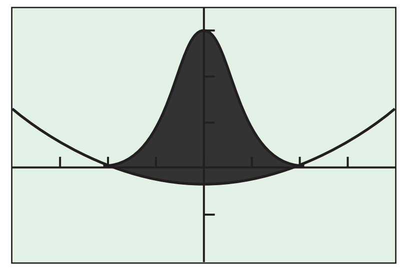
(shown in the next column). ≈1.2956



[–2, 2] by [–1.5, 1.5]

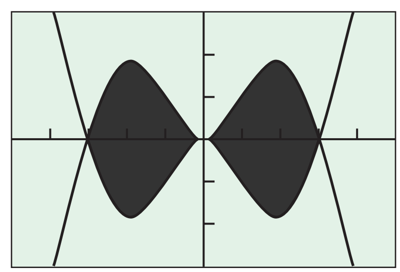
18. ***The Bell*** the region enclosed by the graphs of

$$y = 3^{1-x^2} \quad \text{and} \quad y = \frac{x^2 - 3}{10}$$ ≈5.7312



[–4, 4] by [–2, 3.5]

19. ***The Kissing Fish*** the region enclosed between the graphs of $y = x \sin x$ and $y = -x \sin x$ over the interval $[-\pi, \pi]$ 4π



[–5, 5] by [–3, 3]

20. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = -1$ and $x = 1$ about the x-axis. 2π

21. Find the volume of the solid generated by revolving the region enclosed by the parabola $y^2 = 4x$ and the line $y = x$ about

(a) the x-axis. $32\pi/3$ **(b)** the y-axis. $128\pi/15$

(c) the line $x = 4$. $64\pi/5$ **(d)** the line $y = 4$. $32\pi/3$

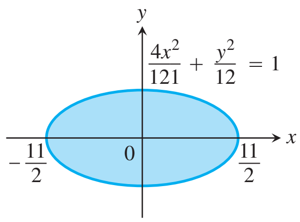
22. The section of the parabola $y = x^2/2$ from $y = 0$ to $y = 2$ is revolved about the y-axis to form a bowl.

(a) Find the volume of the bowl. 4π

(b) Find how much the bowl is holding when it is filled to a depth of k units $(0 < k < 2)$. πk^2

(c) If the bowl is filled at a rate of 2 cubic units per second, how fast is the depth k increasing when $k = 1$? $1/\pi$

23. The profile of a football resembles the ellipse shown here (all dimensions in inches). Find the volume of the football to the nearest cubic inch. $88\pi \approx 276$ in^3



UNITS IN INCHES

24. The base of a solid is the region enclosed between the graphs of $y = \sin x$ and $y = -\sin x$ from $x = 0$ to $x = \pi$. Each cross section perpendicular to the x-axis is a semicircle with diameter connecting the two graphs. Find the volume of the solid. $\pi^2/4$

25. The region enclosed by the graphs of $y = e^{x/2}$, $y = 1$, and $x = \ln 3$ is revolved about the x-axis. Find the volume of the solid generated. $\pi(2 - \ln 3)$

26. A round hole of radius $\sqrt{3}$ feet is bored through the center of a sphere of radius 2 feet. Find the volume of the piece cut out. $28\pi/3$ ft$^3 \approx 29.3215$ ft^3

27. Find the length of the arch of the parabola $y = 9 - x^2$ that lies above the x-axis. ≈ 19.4942

28. Find the *perimeter* of the bow-tie-shaped region enclosed between the graphs of $y = x^3 - x$ and $y = x - x^3$. ≈ 5.2454

29. A particle travels at 2 units per second along the curve $y = x^3 - 3x^2 + 2$. How long does it take to travel from the local maximum to the local minimum? 2.296 sec

30. Group Activity One of the following statements is true for all $k > 0$ and one is false. Which is which? Explain. (a) is true

(a) The graphs of $y = k \sin x$ and $y = \sin kx$ have the same length on the interval $[0, 2\pi]$.

(b) The graph of $y = k \sin x$ is k times as long as the graph of $y = \sin x$ on the interval $[0, 2\pi]$.

31. Let $F(x) = \int_1^x \sqrt{t^4 - 1}\, dt$. Find the *exact* length of the graph of F from $x = 2$ to $x = 5$ without using a calculator. 39

32. *Rock Climbing* A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope weighing 0.8 N/m. How much work will it take to lift

(a) the equipment? 4000 J **(b)** the rope? 640 J

(c) the rope and equipment together? 4640 J

33. *Hauling Water* You drove an 800-gallon tank truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You had started out with a full tank of water, had climbed at a steady rate, and had taken 50 minutes to accomplish the 4750-ft elevation change. Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the summit? Water weighs 8 lb/gal. (Do not count the work done getting you and the truck to the top.) 22,800,000 ft-lb

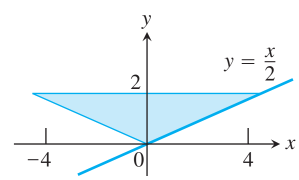
35. No, the work going uphill is positive, but the work going downhill is negative.

34. *Stretching a Spring* If a force of 80 N is required to hold a spring 0.3 m beyond its unstressed length, how much work does it take to stretch the spring this far? How much work does it take to stretch the spring an additional meter? 12 J, ≈ 213.3 J

35. Writing to Learn It takes a lot more effort to roll a stone up a hill than to roll the stone down the hill, but the weight of the stone and the distance it covers are the same. Does this mean that the same amount of work is done? Explain.

36. *Emptying a Bowl* A hemispherical bowl with radius 8 inches is filled with punch (weighing 0.04 pound per cubic inch) to within 2 inches of the top. How much work is done emptying the bowl if the contents are pumped just high enough to get over the rim? ≈ 113.097 in.-lb

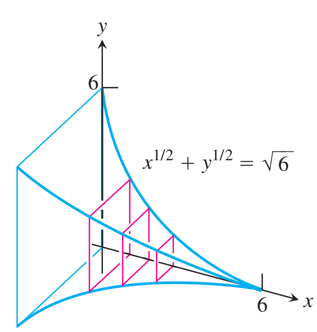
37. *Fluid Force* The vertical triangular plate shown below is the end plate of a feeding trough full of hog slop, weighing 80 pounds per cubic foot. What is the force against the plate? ≈ 426.67 lbs



UNITS IN FEET

38. *Fluid Force* A standard olive oil can measures 5.75 in. by 3.5 in. by 10 in. Find the fluid force against the base and each side of the can when it is full. (Olive oil has a weight-density of 57 pounds per cubic foot.) base ≈ 6.6385 lb, front and back: 5.7726 lb, sides ≈ 9.4835 lb

39. *Volume* A solid lies between planes perpendicular to the x-axis at $x = 0$ and at $x = 6$. The cross sections between the planes are squares whose bases run from the x-axis up to the curve $\sqrt{x} + \sqrt{y} = \sqrt{6}$. Find the volume of the solid. ≈ 14.4



40. *Yellow Perch* A researcher measures the lengths of 3-year-old yellow perch in a fish hatchery and finds that they have a mean length of 17.2 cm with a standard deviation of 3.4 cm. What proportion of 3-year-old yellow perch raised under similar conditions can be expected to reach a length of 20 cm or more? ≈ 0.2051 (20.5%)

41. Group Activity Using as large a sample of classmates as possible, measure the span of each person's fully stretched hand, from the tip of the pinky finger to the tip of the thumb. Based on the mean and standard deviation of your sample, what percentage of students your age would have a finger span of more than 10 inches? Answers will vary.

42. *The 68-95-99.7 Rule* **(a)** Verify that for every normal pdf, the proportion of the population lying within one standard deviation of the mean is close to 68%. (*Hint:* Since it is the same for every pdf, you can simplify the function by assuming that $\mu = 0$ and $\sigma = 1$. Then integrate from -1 to 1.) **(a)** ≈ 0.6827 (68.27%)

(b) Verify the two remaining parts of the rule. **(b)** ≈ 0.9545 (95.45%)

43. Writing to Learn Explain why the area under the graph of a probability density function has to equal 1. The probability that the variable has some value in the range of all possible values is 1.

In Exercises 44–48, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

44. $y = 2x, \quad y = x/2, \quad x = 1$ $\quad \pi$

45. $y = 1/x, \quad y = 0, \quad x = 1/2, \quad x = 2$ $\quad 3\pi$

46. $y = \sin x, \quad y = 0, \quad 0 \le x \le \pi$ $\quad 2\pi^2$

47. $y = x - 3, \quad y = x^2 - 3x$ $\quad 16\pi/3$

48. the bell-shaped region in Exercise 18 $\quad \approx 9.7717$

49. *Bundt Cake* A bundt cake (see Exploration 1, Section 7.3) has a hole of radius 2 inches and an outer radius of 6 inches at the base. It is 5 inches high, and each cross-sectional slice is parabolic.

(a) Model a typical slice by finding the equation of the parabola with y-intercept 5 and x-intercepts ± 2. $\quad y = 5 - \frac{5}{4}x^2$

(b) Revolve the parabolic region about an appropriate line to generate the bundt cake and find its volume. $\quad \approx 335.1032 \text{ in}^3$

50. *Finding a Function* Find a function f that has a continuous derivative on $(0, \infty)$ and that has both of the following properties.

i. The graph of f goes through the point $(1, 1)$.

ii. The length L of the curve from $(1, 1)$ to any point $(x, f(x))$ is given by the formula $L = \ln x + f(x) - 1$.

In Exercises 51 and 52, find the area of the surface generated by revolving the curve about the indicated axis.

51. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x\text{-axis}$ $\quad \approx 3.84$

52. $xy = 1, \quad 1 \le y \le 2; \quad y\text{-axis}$ $\quad \approx 5.02$

50. $f(x) = \dfrac{x^2 - 2\ln x + 3}{4}$

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

53. Let R be the region in the first quadrant enclosed by the y-axis and the graphs of $y = 2 + \sin x$ and $y = \sec x$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

54. The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right), \quad 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

(a) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(b) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(c) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24-hour period?

55. The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{t^2 - 24t + 160}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{t^2 - 38t + 370}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour, and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, which are the hours that the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round your answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?